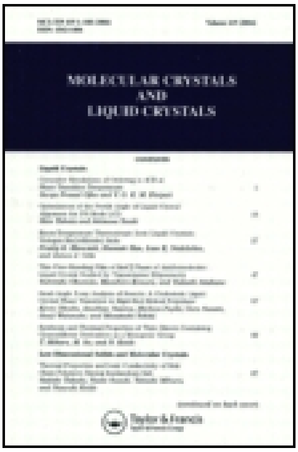

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Compressible Liquid Crystals

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Compressible Liquid Crystals

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The renormalization group is used to study the uniaxial to biaxial transition in liquid crystals. By explicitly writing out the components of the 3 × 3 symmetric traceless tensor, and integrating out the non-critical degrees of freedom, we find a stable fixed point corresponding to an $n = 4$ component magnet. The critical heat exponent for the $n = 4$ system is negative, so the addition of compressibility should not change the order of the phase transition.

1. INTRODUCTION

A nematic liquid crystal is a liquid which has no positional ordering but possesses orientational ordering along an axis.[1,2] A biaxial nematic phase is characterized by orientational order along two different axes. The transition from the uniaxial to biaxial phase has been found to be second order,[3] and X-ray studies reveal that the uniaxial phase is a macroscopic consequence of large orientational fluctuations around the director and only small amplitude oscillations in the biaxial phase.[4,5,6] For a recent overview of biaxial nematics see References 7 and 8.

We are building on the work of Vause and Sak.[9] Vause and Sak (VS) developed the fluctuation theory of the Landau point[10] using the renormalization group[11] and the ε expansion.[12] Our purpose is to consider the transition from the uniaxial phase to the biaxial phase to examine the fixed point behavior starting from mean field theory, (MFT). We start with a rigid lattice and using the renormalization group, find that the critical heat exponent α is negative, so we expect that the addition of compressibility to the system will not change the second order phase transition to a first order transition.

2. MEAN-FIELD THEORY

The following results are from VS,[9] we will expand upon their discussion in subsequent sections. Landau theory neglects the fluctuations, and assumes that the thermodynamic potential can be written in a power series close to a transition where the order parameter vanishes. The coefficients are taken to be analytic functions of the temperature and pressure. One of the first steps in constructing our Landau theory is to choose the order parameter which will represent the liquid crystal system. The order parameter cannot be a scalar since a scalar quantity cannot describe orientational order. It cannot be a vector quantity since there is an inversion symmetry of the director (**n** is equivalent to $-\mathbf{n}$). So we choose a tensor of rank two, which is symmetric and traceless, of order three. We can choose the order parameter to be, in a macroscopic model, fluctuations of the dielectric tensor $\delta\varepsilon_{ij}$, or the magnetic susceptibility $\delta\chi_{ij}$. For instance, the magnetic moment **M** due to molecular diamagnetism if given by $M_\alpha = \chi_{\alpha\beta}H_\beta$, $\alpha, \beta = 1, 2, 3$, where **H** is the magnetic field. In the nematic phase we can write,

$$\chi = \begin{pmatrix} \chi_\perp & 0 & 0 \\ 0 & \chi_\perp & 0 \\ 0 & 0 & \chi_\| \end{pmatrix}$$

To construct an order parameter from the susceptibility we may define $Q_{\alpha\beta} = 1/\delta_{\chi\max}(\chi_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}\chi_{\gamma\gamma})$. A microscopic model can be set up similarly,

$$Q_{\alpha\beta} = \frac{3}{2}x\left(n_\alpha n_\beta - \frac{1}{3}\delta_{\alpha\beta}\right) - \frac{1}{2}y(m_\alpha m_\beta - (\hat{n}\times\hat{m})_\alpha(\hat{n}\times\hat{m})_\beta)$$

where $\hat{n}$, $\hat{m}$, and $\hat{n}\times\hat{m}$ are unit vectors associated with the molecule. In the proper coordinate system, Q can be written as,[8]

$$Q = \begin{bmatrix} x & 0 & 0 \\ 0 & -\frac{1}{2}(x+y) & 0 \\ 0 & 0 & -\frac{1}{2}(x-y) \end{bmatrix}$$

The order parameter Q is a symmetric traceless matrix, which can be written in diagonal form, we find it useful to represent Q in two ways,

$$Q = \begin{pmatrix} \varphi_1+\varphi_2 & 0 & 0 \\ 0 & \varphi_1-\varphi_2 & 0 \\ 0 & 0 & -2\varphi_1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix}$$

It's convenient to use the quantities a and b when we minimize the free energy, where we define,

$$a = \varphi_1 + \varphi_2$$

$$b = \varphi_1 - \varphi_2$$

The properties of Q in the isotropic, nematic and biaxial phases are:

1. *Isotropic*: $\varphi_1 = \varphi_2 = 0$, or in terms of the redefined quantities, $a = b = 0$.

2. *Uniaxial nematic*: $\varphi_1 \neq 0$, $\varphi_2 = 0$, or $a = b \neq 0$. This phase has two different orderings, prolate or rod-like, and oblate or disk-like order. Oblate molecules have a natural tendency to lie with their long axis perpendicular to the director, while the prolate molecules have their long axis parallel to the director.
3. *Biaxial nematic*: $\varphi_1 \neq 0$, $\varphi_2 \neq 0$, or $a \neq b$. There is ordering along the director and ordering along a secondary axis.

We write down the most general free energy equation, keeping the absolute rotational invariants of Q up to sixth order,[8] (note that for a 3×3 matrix $(\mathrm{Tr}Q^2)^2 = 2\mathrm{Tr}Q^4$),

$$F = F_0 = \frac{1}{4} r\mathrm{Tr}Q^2 + u_3\mathrm{Tr}Q^3 + u_4(\mathrm{Tr}Q^2)^2 + u_5(\mathrm{Tr}Q^2)(\mathrm{Tr}Q^3) + u_6(\mathrm{Tr}Q^3)^2 + u_6'(\mathrm{Tr}Q^2)^3 \quad (1)$$

Substituting the expression for Q into the free energy equation we find,

$$F = F_0 + \frac{1}{2} r(a^2 + b^2 + ab) - 3u_3ab(a + b) + 4u_4(a^2 + b^2 + ab)^2 - 6u_5ab(a + b)(a^2 + b^2 + ab) + 9u_6a^2b^2(a + b)^2 + 8u_6'(a^2 + b^2 + ab)^3 \quad (2)$$

We take u_4, u_5, u_6, u_6' to be constants. First, we consider the uniaxial phase, where $a = b$. We minimize F with respect to a and b, (neglecting u_5 and u_6'),

$$\frac{\partial F}{\partial a} = (2a + b)\left[\frac{r}{2} - 3u_3b + 8u_4(a^2 + b^2 + ab) + 18u_6ab^2(a + b)\right] \quad (3)$$

$$\frac{\partial F}{\partial b} = (2b + a)\left[\frac{r}{2} - 3u_3a + 8u_4(a^2 + b^2 + ab) + 18u_6a^2b(a + b)\right] \quad (4)$$

If we also drop u_6 we immediately find the isotropic solution $a = b = 0$. For a and b nonzero, and $a = b$, (the uniaxial phase),

$$a_{\pm} = \frac{1}{16u_4}\left[u_3 \pm \sqrt{u_3^2 - \frac{16}{3} ru_4}\right]$$

Since a must be real, $a_{\pm}$ is a solution when $u_3^2 \geq \frac{16}{3}ru_4$. The coexistence curve between the isotropic and uniaxial phase can be found by letting $F = F_0$,

$$\frac{3}{2} ra^2 - 6u_3a^3 + 36u_4a^4 = 0 \quad (5)$$

We see that either $a = 0$ or,

$$a_{\pm} = \frac{1}{12u_4}[u_3 \pm \sqrt{u_3^2 - 6ru_4}] \tag{6}$$

So a first becomes non-zero when $u_3^2 = 6ru_4$, for $r > 0$. When $u_3^2 > \frac{16}{3}ru_4$ and $u_3 > 0$ the minimum of the free energy occurs when $a = b = a_+$, the prolate phase, and when $u_3 < 0$ the minimum occurs when $a = b = a_-$, the oblate phase, (see Figure B.1). The transition from a uniaxial prolate to uniaxial oblate is first order and can be seen by examining the change in the order parameter as its switches from a_+ to a_-.

To generate a biaxial phase, we will need to keep terms to sixth order. The u_6' term is symmetric in a and b when we minimize it with respect to a or b so u_6' cannot lift the degeneracy between a and b, and we can neglect it.

The u_6 term is the key term to generate biaxial behavior. To see this first consider the free energy term $-u_3ab(a + b)$, (which is negative). This term is minimized when ab is maximized, ($a + b$ is fixed on the uniaxial-biaxial transition curve, $a + b = 2\varphi_1$, with no contribution from the symmetry breaking order parameter φ_2) this occurs when $a = b$, (as in the case of the problem of maximizing the area of a rectangle with a fixed parameter). If $u_6 < 0$, then the u_6 term is minimized when $a = b$, again uniaxial. If $u_6 > 0$, the sixth order energy term is minimized ($a + b$ fixed) when $a = 0$ or $b = 0$, this is the biaxial phase.

To find the mean field values for the order parameters, we subtract both first derivatives of the free energy equation,

$$\frac{1}{2a + b}\frac{\partial F}{\partial a} - \frac{1}{2b + a}\frac{\partial F}{\partial b} = 0 \tag{7}$$

Next we convert from the variables a and b to φ_1 and φ_2, then we find

$$-6u_3\varphi_2 + 72u_6\varphi_1\varphi_2(\varphi_1^2 - \varphi_2^2) = 0$$

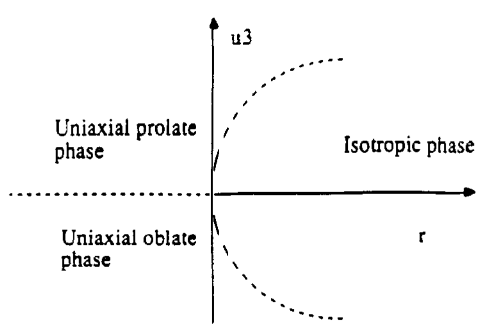

FIGURE B.1 Phase diagram for $u_6 \leq 0$. There are 3 phases, an isotropic phase, uniaxial prolate, and uniaxial oblate. All transitions are first order.

Either $\varphi_2 = 0$, (uniaxial case), or

$$u_3 = 12u_6\varphi_1^3$$

where we have assumed that $\varphi_2^2 << \varphi_1^2$. To find the value of φ_1, we neglect the small u_6 term, and minimize F with respect to φ_1 and obtain,

$$\varphi_1 = \pm\left(\frac{-r}{48u_4}\right)^{1/2} = M$$

So we get the coexistence curve for the uniaxial-biaxial transition, (see Figure B2),

$$u_3 = 12u_6M^3 = \pm 12u_6\left(\frac{-r}{48u_4}\right)^{3/2}$$

The addition of a small u_5 term, changes the coexistence curve by $-u_5(-r/8u_4)$. VS[9] has found that u_5 'rotates' the biaxial cusp, while not changing the topological nature of the phase diagram. Since is does not change the critical behavior of the system we will neglect it from here on.

We will examine the point along the second order transition curve, from the uniaxial nematic to biaxial nematic phase, far enough from the Landau point, ($r \sim -1$) so that fluctuations about the director can be considered small.

3. RECURSION RELATIONS

Since we are looking at a second order phase transition we redefine our variables to reflect the fact that the φ_1 is non-zero along the U-B transition so that we may deal only with vanishing order parameters.

We separate the order parameter into diagonal and off-diagonal components,

$$Q_{ij} = S_i\delta_{ij} + \hat{Q}_{ij}$$

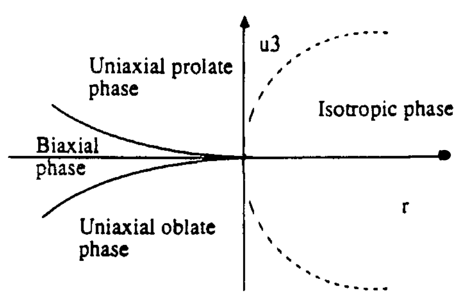

FIGURE B.2 Phase diagram for $u_6 > 0$. There are four phases, a isotropic phase, a uniaxial prolate, a uniaxial oblate and a biaxial phase. The transition between the uniaxial phases and the biaxial phase is second order, as is the Landau point, at ($r = 0$, $u_3 = 0$). We are interested in examining a point along the uniaxial-biaxial transition curve, far from the Landau point, $r \sim -1$.

where the diagonal elements are given by,

$$S_1 = M + \varphi_1 + \varphi_2$$

$$S_2 = M + \varphi_1 - \varphi_2$$

$$S_3 = -2M - 2\varphi_1$$

and the off-diagonal elements are,

$$\hat{Q}_{ij} = \begin{pmatrix} 0 & \hat{Q}_{12} & \hat{Q}_{13} \\ \hat{Q}_{12} & 0 & \hat{Q}_{23} \\ \hat{Q}_{13} & \hat{Q}_{23} & 0 \end{pmatrix}$$

Here, M is the non-zero expectation value of our original φ_1. We then define the quantity $\langle M + \varphi_1 \rangle$ as the expectation value of the director in the uniaxial phase, with φ_1 being a variable which fluctuates about zero. Our liquid crystal Hamiltonian is,

$$H = \int d^dx \left[\frac{1}{4} r \mathrm{Tr} Q^2 + \frac{1}{4} \nabla_k Q_{ij} \nabla_k Q_{ij} + u_3 \mathrm{Tr} Q^3 + u_4 (\mathrm{Tr} Q^2)^2 + u_6 (\mathrm{Tr} Q^3)^2 \right] \quad (8)$$

Then we expand the Hamiltonian in terms of the fluctuating quantities, this can be done diagrammatically, as shown in Figure B3.

The resulting Hamiltonians are,

$$H = H_c + H_0 + H_1 + H_3 + H_4$$

$$H_c = L^d \left[\frac{3}{2} r M^2 - 6 u_3 M^3 + 36 u_4 M^4 + 36 u_6 M^6 \right]$$

$$H_0 = \int d^dx \left[\frac{3}{2} r_1 \varphi_1^2 + \frac{1}{2} r_2 \varphi_2^2 + \frac{1}{2} r_3 \hat{Q}_{12}^2 + \frac{1}{2} r_4 \hat{Q}_{13}^2 + \frac{1}{2} r_5 \hat{Q}_{23}^2 + \frac{1}{4} \nabla_k Q_{ij} \nabla_k Q_{ij} \right]$$

$$H_1 = h \int d^dx \varphi_1(x)$$

$$H_3 = \int d^dx [\varphi_1 (y_1 \varphi_2^2 + y_2 \hat{Q}_{12}^2 + y_3 \hat{Q}_{13}^2 + y_4 \hat{Q}_{23}^2) + y_5 \varphi_2 \hat{Q}_{13}^2 + y_6 \varphi_2 \hat{Q}_{23}^2 + y_7 \hat{Q}_{12} \hat{Q}_{13} \hat{Q}_{23}]$$

$$H_4 = \int d^dx [z_1 \varphi_2^4 + z_2 \varphi_2^2 \hat{Q}_{12}^2 + z_3 \varphi_2^2 \hat{Q}_{13}^2 + z_4 \varphi_2^2 \hat{Q}_{23}^2 + z_5 \hat{Q}_{12}^4 + z_6 \hat{Q}_{13}^4 + z_7 \hat{Q}_{23}^4 + z_8 \hat{Q}_{12}^2 \hat{Q}_{13}^2 + z_9 \hat{Q}_{12}^2 \hat{Q}_{23}^2 + z_{10} \hat{Q}_{13}^2 \hat{Q}_{23}^2]$$

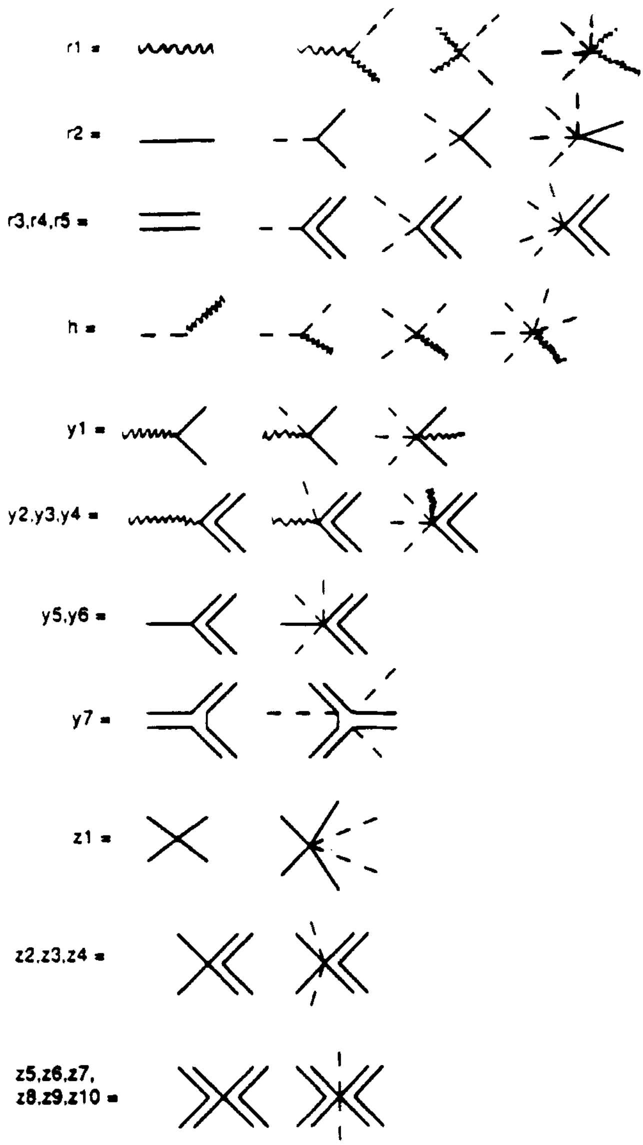

FIGURE B.3 Graphical representation of the coefficients after shifting φ_1 by M. The dotted line is M, while the wavy line is φ_1. The straight line represents φ_2, and the double lines are the off-diagonal elements $\hat{Q}_{12}$, $\hat{Q}_{13}$, and $\hat{Q}_{23}$.

where L is the length, d is the dimension of the system and $M = \pm(-r/48u_4)^{1/2}$. The quadratic coefficients are given as,

$$r_1 = r - 12u_3M + 144u_4M^2 + 360u_6M^4 \tag{9}$$

$$r_2 = r + 12u_3M + 48u_4M^2 - 144u_6M^4 \tag{10}$$

$$r_4 = r - 6u_3M + 48u_4M^2 + 76u_6M^4 \tag{11}$$

where $r_2 = r_3$, and $r_4 = r_5$. We note that r_2, r_3, r_4, and r_5 vanish on the U-B transition curve. This is a crucial result since this means that we have 4 critical components that determine the behavior of the system.

Since the coefficient $r_1 = -2r + 216u_6M^4$, $u_6 > 0$ and $r < 0$ on the uniaxial-biaxial coexistence curve, we see that $r_1 > 0$. This implies that we can consider φ_1 to have Gaussian fluctuations and that we only need to keep terms which are quadratic in φ_1 and those which are linear. However we must keep terms which are of fourth order in φ_2 and the off-diagonal elements for thermodynamic stability.

h is the coefficient of the linear term, and it vanishes on the U-B transition curve,

$$h = 3rM - 18u_3M^2 + 144u_4M^3 + 216u_6M^5 \tag{12}$$

The next coefficients are associated with the third order terms in the Hamiltonian,

$$y_1 = 6u_3 + 48u_4M - 288u_6M^3 \tag{13}$$

$$y_3 = -3u_3 + 48u_4M + 144u_6M^3 \tag{14}$$

$$y_5 = 3u_3 - 36u_6M^3 \tag{15}$$

The rest of the third order coefficients are given by the relations, $y_1 = y_2$, $y_3 = y_4$, and $y_5 = -y_6 = \frac{1}{2}y_7$.

The last set of quartic coefficients are given by,

$$z_1 = 4u_4 + 36u_6M^2 \tag{16}$$

$$z_3 = 8u_4 - 36u_6M^2 \tag{17}$$

$$z_6 = 4u_4 + 9u_6M^2 \tag{18}$$

and $z_1 = \frac{1}{2}z_2 = z_5$, $z_3 = z_4 = z_8 = z_9$, and $z_6 = z_7 = \frac{1}{2}z_{10}$.

To begin our momentum-shell renormalization group procedure, we first Fourier transform the variables by letting

$$Q_{ij}(\mathbf{x}) = L^{-d} \sum_{\mathbf{k}} Q_{ij}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$$

The φ_1 variables can be integrated immediately since they undergo Gaussian fluctuations. The integration, which involves completing the square is shown below,

$$\exp[-H_{\text{eff}}] = \exp[-(H_0' + H_3' + H_4)]$$

$$\times \int \mathcal{D}\varphi_1 \exp\left[-\frac{3}{2}L^{-d}\sum_k \left((r_1 + k^2)\varphi_1(k)\varphi_1(-k)\right.\right.$$

$$+ \frac{2}{3}\varphi_1(k)g(-k)\bigg)\bigg]$$

$$= \exp[-(H_0' + H_3' + H_4)] \int \mathfrak{D}\varphi_1 \exp\left[-\frac{3}{2}L^{-d}\sum_k (r_1 + k^2)\right.$$

$$\left.\times \left(\varphi_1(k) + \frac{1}{3}\frac{g(k)}{r_1 + k^2}\right)\left(\varphi_1(-k) + \frac{1}{3}\frac{g(-k)}{r_1 = k^2}\right)\right]$$

$$\times \exp\left[\frac{1}{6}L^{-d}\sum_k \frac{1}{r_1 + k^2} g(k)g(-k)\right]$$

where we have renamed the Hamiltonians unaffected by the φ_1 integration,

$$H_0' = \frac{1}{2}L^{-d}\sum_k (r_2 + k^2)\varphi_2(k)\varphi_2(-k) + \frac{1}{2}L^{-d}$$

$$\cdot\sum_k (r_3 + k^2)\,\hat{Q}_{12}(k)\hat{Q}_{12}(-k)$$

$$+ \frac{1}{2}L^{-d}\sum_k (r_4 + k^2)\hat{Q}_{13}(k)\hat{Q}_{13}(-k) + \frac{1}{2}L^{-d}$$

$$\cdot\sum_k (r_5 + k^2)\hat{Q}_{23}(k)\hat{Q}_{23}(-k)$$

$$H_3' = L^{-2d}\sum_{k_1,k_2,k_3} \delta(k_1 + k_2 + k_3)[y_5\varphi_2(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{13}(k_3)$$

$$+ y_6\varphi_2(k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(k_3) + y_7\hat{Q}_{12}(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{23}(k_3)]$$

$$g(-k) = h\delta(-k)L^d + L^{-d}\sum_{k_1,k_2} \delta(k + k_1 + k_2)[y_1\varphi_2(k_1)\varphi_2(k_2)$$

$$+ y_2\hat{Q}_{12}(k_1)\hat{Q}_{12}(k_2) + y_3\hat{Q}_{13}(k_1)\hat{Q}_{13}(k_2) + y_4\hat{Q}_{23}(k_1)\hat{Q}_{23}(k_2)] \quad ;)]$$

The integrals over φ_1 yield constants. If we expand the last term,

$$\exp\left[\frac{1}{6}L^{-d}\sum_k \frac{1}{r_1 + k^2} g(k)g(-k)\right]$$

the effective Hamiltonian becomes,

$$H_{\text{eff}} = \frac{1}{2}L^{-d}\sum_k (\tau_2 + k^2)\varphi_2(k)\varphi_2(-k)$$

$$+ \frac{1}{2}L^{-d}\sum_k (\tau_3 + k^2)\hat{Q}_{12}(k)\hat{Q}_{12}(-k)$$

$$+ \frac{1}{2} L^{-d} \sum_k (\tau_4 + k^2)\hat{Q}_{13}(k)\hat{Q}_{13}(-k)$$

$$+ \frac{1}{2} L^{-d} \sum_k (\tau_5 + k^2)\hat{Q}_{23}(k)\hat{Q}_{23}(-k)$$

$$+ L^{-2d} \sum_{k_1,k_2,k_3} \delta(k_1 + k_2 + k_3)[y_5\varphi_2(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{13}(k_3)$$

$$+ y_6\varphi_2(k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(k_3) + y_7\hat{Q}_{12}(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{23}(k_3)]$$

$$+ L^{-3d} \sum_{k_1,k_2,k_3,k_4} \delta(k_1 + k_2 + k_3 + k_4)$$

$$\times [v_1(k_1 + k_2)\varphi_2(k_1)\varphi_2(k_2)\varphi_2(k_3)\varphi_2(k_4)$$

$$+ v_2(k_1 + k_2)\varphi_2(k_1)\varphi_2(k_2)\hat{Q}_{12}(k_3)\hat{Q}_{12}(k_4)$$

$$+ v_3(k_1 + k_2)\varphi_2(k_1)\varphi_2(k_2)\hat{Q}_{13}(k_3)\hat{Q}_{13}(k_4)$$

$$+ v_4(k_1 + k_2)\varphi_2(k_1)\varphi_2(k_2)\hat{Q}_{23}(k_3)\hat{Q}_{23}(k_4)$$

$$+ v_5(k_1 + k_2)\hat{Q}_{12}(k_1)\hat{Q}_{12}(k_2)\hat{Q}_{12}(k_3)\hat{Q}_{12}(k_4)$$

$$+ v_6(k_1 + k_2)\hat{Q}_{13}(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{13}(k_3)\hat{Q}_{13}(k_4)$$

$$+ v_7(k_1 + k_2)\hat{Q}_{23}(k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(k_3)\hat{Q}_{23}(k_4)$$

$$+ v_8(k_1 + k_2)\hat{Q}_{12}(k_1)\hat{Q}_{12}(k_2)\hat{Q}_{13}(k_3)\hat{Q}_{13}(k_4)$$

$$+ v_9(k_1 + k_2)\hat{Q}_{12}(k_1)\hat{Q}_{12}(k_2)\hat{Q}_{23}(k_3)\hat{Q}_{23}(k_4)$$

$$+ v_{10}(k_1 + k_2)\hat{Q}_{13}(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{23}(k_3)\hat{Q}_{23}(k_4)]$$

The coefficients are given in the appendix.

We perform the renormalization group[11] by first rewriting each of the order parameters $Q_{ij}(x) = L^{-d}(\Sigma_{k<b^{-1}} Q_{ij}e^{i\mathbf{k}\cdot\mathbf{x}} + \Sigma_{b^{-1}\leq k\leq 1} Q_{ij}e^{i\mathbf{k}\cdot\mathbf{x}})$, $(b > 1)$, then we integrate out the degrees of freedom over all $\hat{Q}_{ij}$ and φ_2 such that $b^{-1} \leq k \leq 1$, rescale the order parameters by ζ, $(\zeta^2 = b^{d+2-\eta})$, change the length scale using the primed quantities $L = bL'$, and $k = k'/b$. This can be done diagrammatically as shown in Figure B.4 below, (see Reference 13). We will follow the traditional convention and choose η so that the coefficient of k^2 remains unchanged under this transformation, $(\eta = O(\varepsilon^2))$. Finally, integrate the Feynman integrals of the form $A(r) = \int_{\mathbf{k}}^{>} 1/(r + k^2)$ and $B(r) = \int_{\mathbf{k}}^{>} [1/(r + k^2)]^2$, where $\int_{\mathbf{k}}^{>} = \int_{b^{-1}\leq k\leq 1} d^dk$.

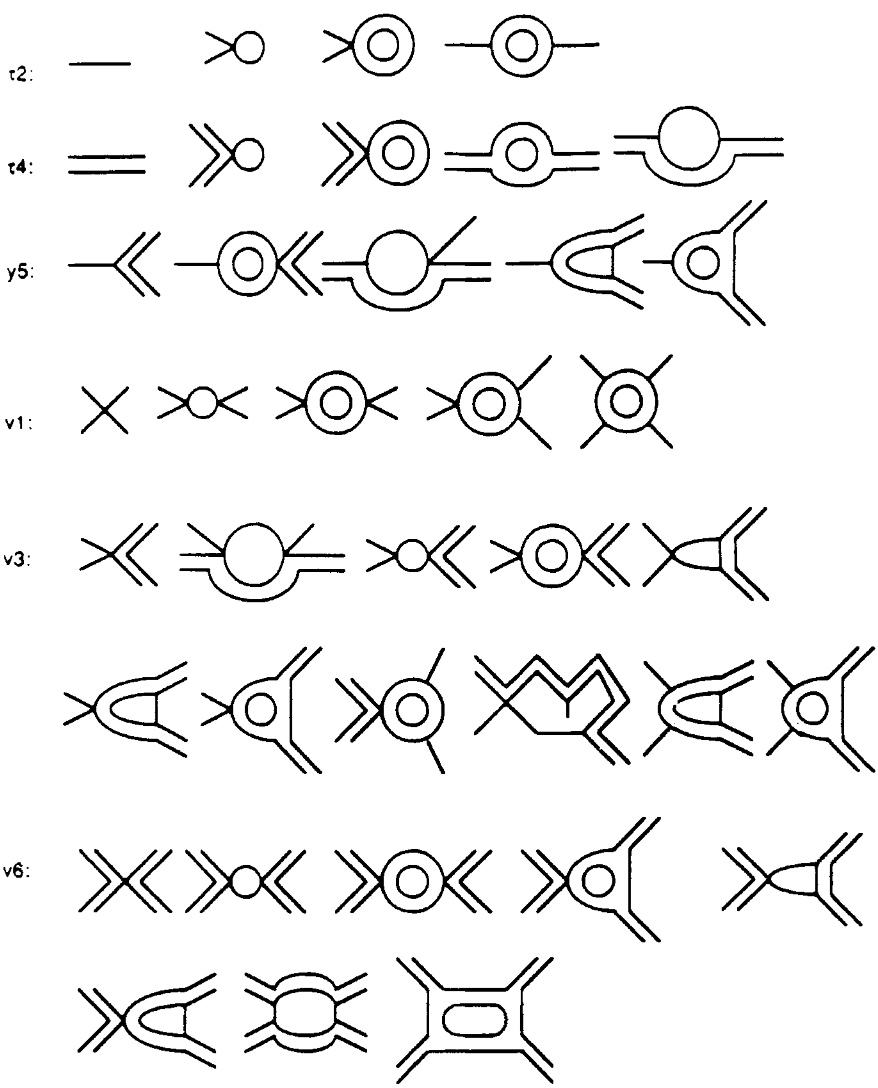

FIGURE B.4 Diagrammatic representation of the recursion relations. The straight lines represent φ_2, and the double lines represent either $\hat{Q}_{12}$, $\hat{Q}_{13}$, or $\hat{Q}_{23}$.

Since the quadratic and quartic terms are momentum dependent, the recursion relations are nonlinear functional equations,[11] but we can justify writing these as algebraic equations by noting that we are only interested in the macroscopic effects, (small momentum), so we may consider the important wavelengths k^2 to be of order ε, which are small compared to r_1. We then compute the differential recursion relations by letting $b = e^{\delta l}$, δl infinitesimal.

We find that the equalities among the quadratic and quartic coefficients which hold in MFT also hold under the differential recursion relations, this means that starting from the mean field solutions, the relations $\tau_2 = \tau_3$, $\tau_4 = \tau_5$ . . . , will continue to be satisfied for all values of $b > 1$. So we only need to consider 2 of the 4 quadratic coefficients, 1 of the 3 third order coefficients, and 3 out of 10 fourth order coefficients.

Our final set of recursion relations are,

$$\tau_2' = b^2\{\tau_2 + K_4[8v_1(1 - b^{-2} - 2\tau_2 \ln b) + 2v_3(1 - b^{-2} - 2\tau_4 \ln b)]\} \quad (19)$$

$$\tau_4' = b^2\{\tau_4 + K_4[8v_6(1 - b^{-2} - 2\tau_4 \ln b) + 2v_3(1 - b^{-2} - 2\tau_2 \ln b)]\} \quad (20)$$

$$y_5' = b^{1+(\varepsilon/2)}[y_5 - K_4(8y_5v_3 \ln b + 8y_5v_6 \ln b)] \quad (21)$$

$$v_1' = b^\varepsilon[v_1 - K_4(40v_1^2 \ln b + 2v_3^2 \ln b)] \quad (22)$$

$$v_3' = b^\varepsilon[v_3 - K_4(8v_3^2 \ln b + 16(v_1 + v_6)v_3 \ln b)] \quad (23)$$

$$v_6' = b^\varepsilon[v_6 - K_4(40v_6^2 \ln b + 2v_3^2 \ln b)] \quad (24)$$

where $K_4 = 1/8\pi^2$.

When $v_3^* = 0$, the linearized renormalization group matrix is upper triangular, so the critical exponents can be found directly from the diagonal elements of the linearized matrix,

$$\lambda_{\tau 2} = 2 - 16K_4v_1^* \quad (25)$$

$$\lambda_{\tau 4} = 2 - 16K_4v_6^* \quad (26)$$

$$\lambda_{y5} = 1 + \frac{\varepsilon}{2} - 8K_4(v_3^* + v_6^*) \quad (27)$$

$$\lambda_{v1} = \varepsilon - 80K_4v_1^* \quad (28)$$

$$\lambda_{v3} = \varepsilon - 16K_4(v_1^* + v_3^* + v_6^*) \quad (29)$$

$$\lambda_{v6} = \varepsilon - 80K_4v_6^* \quad (30)$$

At the fixed point $F5$, $\tau_2 = \tau_4 = -\varepsilon/4$, $y_5 = 0$, $v_1 = v_6 = 2\pi^2\varepsilon/12$, and $v_3 = 2\pi^2\varepsilon/6$. The values of the critical exponents to first order in ε are,

$$\lambda_{1,2} = 2 - \frac{\varepsilon}{3} \mp \frac{\varepsilon}{6} \quad (31)$$

$$\lambda_{y5} = 1 \quad (32)$$

$$\lambda_3 = -\frac{2\varepsilon}{3} \quad (33)$$

$$\lambda_{4,5} = -\frac{\varepsilon}{2} \pm \frac{\varepsilon}{2} \tag{34}$$

The first four fixed points represent decoupled Hamiltonians, similar to bicritical behavior,[14–16] with $n_{\parallel} = n_{\perp} = 2$. We find that these recursion relations and fixed points correspond to the anisotropic antiferromagnet in a uniform field, when both the anisotropy and field single out $m_1 = 2$ spin components, $m_2 = n - m_1 = 2$, where,[17]

$$\bar{H} = \int_{\mathbf{x}} \left\{ \frac{1}{2} [r_1 \mathbf{S}_1^2 + r_2 \mathbf{S}_2^2 + (\nabla \mathbf{S}_1)^2 + (\nabla \mathbf{S}_2)^2] + u|\mathbf{S}_1|^4 + v|\mathbf{S}_2|^4 + 2w|\mathbf{S}_1|^2|\mathbf{S}_2|^2 \right\} \tag{35}$$

Fixed point $F1$ is the Gaussian fixed point. Fixed point $F2$ corresponds to the decoupled $n = 2$ spin Hamiltonian used by Jacobsen and Swift,[19] however, we find that the fixed points corresponding to the decoupled $n = 2$ critical behavior is unstable. Fixed points $F2$ and $F3$ are called the 1 and 2 phase, while $F4$ is called the mixed phase.[20] The above fixed points are unstable in v_3, so we expect that the critical behavior will not be governed by these fixed points. The last fixed point, $F5$, corresponds to the $n = 4$ component magnet, it is stable and accessible, so we conclude that the behavior of the transition from uniaxial to biaxial nematic liquid crystal is a second order transition.

The critical heat exponent α can be found from the hyperscaling relation,

$$\alpha = 2 - d\nu \tag{36}$$

Here, $1/\nu = 2 - \frac{1}{2}\varepsilon$, and at $d = 4 - \varepsilon$ we find,

$$\alpha = 0 + O(\varepsilon^2) \tag{37}$$

This is in agreement with the critical heat exponent calculated using the n-component vector model, when $n = 4$,

$$\alpha = 0 + \frac{4 - n}{2(n + 8)} \varepsilon - \frac{(n + 2)^2(n + 28)}{4(n + 8)^3} \varepsilon^2 + \ldots$$

Since α is non-positive for $n = 4$, the addition of compressibility should not change the order of the transition.

4. COMPRESSIBLE LIQUID CRYSTALS

There have been many early studies of compressible n-component magnets,[21] as well as those using the modern renormalization group methods.[17,18,22–25] We will consider our model[9,26] to be an elastically isotropic liquid crystal with free surfaces, (Wegner[27]) has shown that surface deformations do not affect the critical behavior), and will neglect shear forces, but we have to be careful when doing the Gaussian integrals to take the limit as the shear modulus μ goes to zero, since there are factors independent of the order parameter which diverge in this limit. Sak[23] found that the order of the transition of the n-component compressible magnet was determined by the sign of the critical heat exponent of the n-component rigid magnet, α. We would like to investigate the effects of compressibility on a liquid crystal system, specifically, considering the uniaxial-biaxial transition, and noting that the exponent α was found to be non-positive for the second order U-B transition. To do this, we start with the liquid crystal Hamiltonian of Vause and Sak,[9] (see also Reference 26) adding in the isotropic elastic Hamiltonian as well as the coupling between the tensor energy and the elastic degrees of freedom.

Our starting Hamiltonian is,

$$H = H_l + H_e + H_{el} + H_{pr} \tag{38}$$

$$H_l = \int d^dx\left[\frac{1}{4}(r_0\text{Tr}Q^2 + \nabla_k Q_{ij}\nabla_k Q_{ij}) + u_3\text{Tr}Q^3 + u_4^0(\text{Tr}Q^2)^2 + u_6(\text{Tr}Q^3)^2\right] \tag{39}$$

$$H_e = \int d^dx\left[\left(\frac{1}{2}K - \frac{1}{d}\mu\right)[\nabla\cdot\mathbf{u}(\mathbf{x})]^2 + \mu\sum_{\alpha\beta=1}^{d}\left(\frac{\partial u_\alpha}{\partial x_\beta}\right)^2\right] \tag{40}$$

$$H_{el} = g\int d^dx\text{Tr}Q^2[\nabla\cdot\mathbf{u}(\mathbf{x})] \tag{41}$$

$$H_{pr} = PL^d\left(1 + \sum_{\alpha=1}^{d} e_{\alpha\alpha} + \sum_{\alpha\neq\beta} e_{\alpha\alpha}e_{\beta\beta}\right) \tag{42}$$

H_l is the tensor energy[9,26] of the incompressible liquid crystal. r_0 and u_3 are analytic functions of the thermodynamic fields, u_4 and u_6 are small constants. $Q_{ij} = Q_{ij}(\mathbf{x})$ is a traceless, symmetric tensor of order 3. We will ignore terms such as $\nabla_i Q_{ij}\nabla_k Q_{kj}$ which are analogous to the dipole term in the vector model.[26] H_e is the energy of a deformed elastic medium in the harmonic approximation where K and μ are the bulk and shear modulus of the underlying lattice divided by the temperature T, ($\mu \ll K$).[28] For a liquid, μ is zero, but we cannot simply set μ to zero since there are constant terms which appear after computing the Gaussian integrals and they diverge in the $\lim_{\mu\to 0}$. So we must define our averages as,

$$\langle\hat{O}\rangle = \lim_{\mu\to 0} Z^{-1}\text{Tr}(\hat{O}e^{-H}) \tag{43}$$

where $\hat{O}$ is any operator, $H = H_l + H_e + H_{el} + H_{pr}$, and $Z = \text{Tr}(e^{-H})$. The field $\mathbf{u}(\mathbf{x})$ is a d-component vector displacement field. H_{el} is the coupling between the local dilation and the tensor-energy with coupling strength g. H_{pr} takes into account the change in energy of the system when it undergoes a uniform deformation of magnitude $e_{\alpha\alpha}$. First we integrate out the elastic degrees of freedom by defining

$$e^{-H_{\rm eff}} \equiv {\rm Tr}_{\mathbf{u}(\mathbf{x})} e^{-H} \tag{44}$$

Taking care to separate the $k = 0$ part of the deformation from the phonons, we Fourier transform our quantities as follows,

$$\frac{\partial u_\alpha(\mathbf{x})}{\partial x_\beta} = e_{\alpha\beta} + L^{-d} \sum_{\mathbf{k}} ik_\beta u_\alpha(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{45}$$

$$Q_{mn}(\mathbf{x}) = L^{-d} \sum_{\mathbf{k}} Q_{mn}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{46}$$

To perform the Gaussian integrations we first consider the integrals over the zero-wavelength modes. The integrals over $e_{\alpha\beta}$, where $\alpha \neq \beta$ yield constants to $H_{\rm eff}$ and so will be ignored. When $\alpha = \beta$, the integrals contribute to the effective Hamiltonian, and are calculated below. To do these integrals we diagonalize the matrix $B_{\alpha\beta}$ by choosing a set of uniform deformation variables $e_{\alpha\alpha} = \Sigma_\gamma A_{\alpha\gamma} e'_{\gamma\gamma}$, such that $A_{\alpha\gamma} B_{\alpha\beta} A_{\beta\nu} = \delta_{\gamma\nu}$. And, once we have diagonalized the quadratic expression, we can complete the square and integrate the standard Gaussian form,

$$\int de_{\alpha\alpha} \exp\left[-\left(\sum_{\alpha\beta} B_{\alpha\beta} e_{\alpha\alpha} e_{\beta\beta} + (gQ^2 + PL^d) \sum_{\alpha} e_{\alpha\alpha}\right)\right]$$

$$= D^{-1/2} \pi^{d/2} \exp\left[\frac{1}{4}(g\langle Q^2\rangle + PL^d)^2 \sum_{\alpha\beta} B^{-1}_{\alpha\beta}\right] \tag{47}$$

We define,

$$B_{\alpha\beta} = L^d\left((\mu - P)\delta_{\alpha\beta} + \frac{1}{2}K - \frac{1}{d}\mu + P\right) \tag{48}$$

$$B^{-1}_{\alpha\beta} = L^{-d}\left[\frac{1}{\mu - P}\delta_{\alpha\beta} - \frac{1}{d(\mu - P)} \frac{\frac{1}{2}K - \frac{1}{d}\mu + P}{\frac{1}{2}K + \frac{d-1}{d}P}\right] \tag{49}$$

$$D = \det(B) \tag{50}$$

$$Q^2 = L^{-d} \sum_{\mathbf{k}} Q_{ij}(\mathbf{k}) Q_{ij}(-\mathbf{k}) \tag{51}$$

We can sum over the indices αβ to simplify the $\Sigma_{\alpha\beta} B^{-1}_{\alpha\beta}$ term,

$$\sum_{\alpha\beta} B^{-1}_{\alpha\beta} = L^{-d} \frac{1}{\frac{1}{2}K + \frac{d-1}{d}P} \tag{52}$$

Substituting back into the integral,

$$\int \prod_{\alpha} de_{\alpha\alpha} \exp\left[-\left(\sum_{\alpha\beta} B_{\alpha\beta} e_{\alpha\alpha} e_{\beta\beta} + (gQ^2 + PL^d)\sum_{\alpha} e_{\alpha\alpha}\right)\right]$$

$$= D^{-1/2}\pi^{d/2} \exp\left[\frac{1}{4}\frac{L^{-d}}{\frac{1}{2}K + \frac{d-1}{d}P}(gQ^2 + PL^d)^2\right] \tag{53}$$

The phonon integrals are,

$$\int \prod_{\mathbf{k}\leq 1} du_{\alpha\mathbf{k}}\, du_{\alpha-\mathbf{k}} \exp\left[-\left(\sum_{\alpha\beta} B_{\alpha\beta}(\mathbf{k}) u_{\alpha\mathbf{k}} u_{\beta-\mathbf{k}} + gL^{-2d}\sum_{\alpha} G_{\alpha}(\mathbf{k}) u_{\alpha\mathbf{k}}\right)\right]$$

$$\sim \exp\left[\frac{1}{4} g^2 L^{-4d} \sum_{\mathbf{k}} \sum_{\alpha\beta} B^{-1}_{\alpha\beta} k_\alpha k_\beta \sum_{k_1,k_2,k_3,k_4} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)\right.$$

$$\left.\times \delta(-\mathbf{k} + \mathbf{k}_3 + \mathbf{k}_4) Q_{ij}(\mathbf{k}_1) Q_{ij}(\mathbf{k}_2) Q_{mn}(\mathbf{k}_3) Q_{mn}(\mathbf{k}_4)\right] \tag{54}$$

where we have set,

$$G_\alpha(\mathbf{k}) = ik_\alpha \sum_{ij=1}^{3} \sum_{\mathbf{k}_1\mathbf{k}_2} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) Q_{ij}(\mathbf{k}_1) Q_{ij}(\mathbf{k}_2)$$

$$B_{\alpha\beta} = L^{-d}\left(\mu k^2 \delta_{\alpha\beta} + \left(\frac{1}{2}K - \frac{1}{d}\mu\right) k_\alpha k_\beta\right)$$

$$B^{-1}_{\alpha\beta} = \frac{L^d}{\mu k^2}\left(\delta_{\alpha\beta} - \frac{\frac{1}{2}K - \frac{1}{d}\mu}{\frac{1}{2}K + \frac{d-1}{d}\mu}\frac{k_\alpha k_\beta}{k^2}\right)$$

If we interchange the sum over αβ with $\mathbf{k}$, we see that the sum over the indices simplifies our expression to,

$$\sum_{\alpha\beta} B_{\alpha\beta}^{-1} k_\alpha k_\beta = L^d \frac{1}{\frac{1}{2}K + \frac{d-1}{d}\mu} \tag{55}$$

which is $\mathbf{k}$ independent. We then add and subtract a $k = 0$ part, so that we can sum over $\mathbf{k}$ to remove one of the delta functions,

$$\int \prod du_{\alpha\mathbf{k}} du_{\alpha-\mathbf{k}} \exp\left[-\sum_{\mathbf{k}}\left(\sum_{\alpha\beta} B_{\alpha\beta}(\mathbf{k}) u_{\alpha\mathbf{k}} u_{\beta-\mathbf{k}} + gL^{-2d}\sum_{\alpha} G_\alpha(\mathbf{k}) u_{\alpha\mathbf{k}}\right)\right]$$

$$\sim \exp\left[\frac{1}{4} g^2 L^{-3d} \frac{1}{\frac{1}{2}K + \frac{d-1}{d}\mu} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)\right.$$

$$\left. \times\ Q_{ij}(\mathbf{k}_1) Q_{ij}(\mathbf{k}_2) Q_{mn}(\mathbf{k}_3) Q_{mn}(\mathbf{k}_4)\right] \exp\left[-\frac{1}{4} g^2 L^{-3d} \frac{1}{\frac{1}{2}K + \frac{d-1}{d}\mu}\right.$$

$$\left. \sum_{\mathbf{k}} Q_{ij}(\mathbf{k}) Q_{ij}(-\mathbf{k}) \sum_{\mathbf{k}'} Q_{mn}(\mathbf{k}') Q_{mn}(-\mathbf{k}')\right]$$

Having done the Gaussian integrations we obtain the effective Hamiltonian,

$$H_{\text{eff}} = \frac{1}{4} L^{-d} \sum_{\mathbf{k}} (r + k^2) Q_{ij}(\mathbf{k}) Q_{ij}(-\mathbf{k})$$

$$+ u_3 L^{-2d} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) Q_{ij}(\mathbf{k}_1) Q_{jm}(\mathbf{k}_2) Q_{mi}(\mathbf{k}_3)$$

$$+ u_4 L^{-3d} \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)$$

$$\times\ Q_{ij}(\mathbf{k}_1) Q_{ij}(\mathbf{k}_2) Q_{mn}(\mathbf{k}_3) Q_{mn}(\mathbf{k}_4)$$

$$+ wL^{-3d} \sum_{\mathbf{k}} Q_{ij}(\mathbf{k}) Q_{ij}(-\mathbf{k}) \sum_{\mathbf{q}} Q_{mn}(\mathbf{q}) Q_{mn}(-\mathbf{q})$$

$$+ u_6 L^{-5d} \sum_{\mathbf{k}1 \ldots \mathbf{k}6} \delta(\mathbf{k}_1 + \ldots + \mathbf{k}_6) Q_{ij}(\mathbf{k}_1) Q_{jm}(\mathbf{k}_2) Q_{ml}(\mathbf{k}_3)$$

$$\times\ Q_{nl}(\mathbf{k}_4) Q_{lp}(\mathbf{k}_5) Q_{pn}(\mathbf{k}_6)$$

where the coefficients are modified as follows,

$$r = r_0 - \frac{gP}{2} \frac{1}{\frac{1}{2}K + \frac{d-1}{d}P} \tag{56}$$

$$u_4 = u_4^0 - \frac{1}{4} \frac{g^2}{\frac{1}{2}K + \frac{d-1}{d}\mu} \tag{57}$$

$$w = \frac{g^2}{4}\left[\frac{1}{\frac{1}{2}K + \frac{d-1}{d}\mu} - \frac{1}{\frac{1}{2}K + \frac{d-1}{d}P}\right] \tag{58}$$

Since $P > \mu$ this means that $w > 0$. Note that w is the coefficient of a long range energy density-energy density coupling term which is generated in a similar way in compressible magnets.

5. MEAN-FIELD THEORY

With the same order parameter Q as before, with the free energy equation now modified by the presence of an extra fourth order term, we have,

$$F = F_0 + \frac{1}{4} r\mathrm{Tr}Q^2 + u_3\mathrm{Tr}Q^3 + (u_4 + w)(\mathrm{Tr}Q^2) + u_6(\mathrm{Tr}Q^3)^2$$

$$Q = \begin{pmatrix} \varphi_1 + \varphi_2 & 0 & 0 \\ 0 & \varphi_1 - \varphi_2 & 0 \\ 0 & 0 & -2\varphi_1 \end{pmatrix} \tag{59}$$

The analysis follows exactly as we have done previously. Since we are in a mean field theory, we can replace the long range energy density-energy density coupling with its mean field values, in other words, $\int d^dx\mathrm{Tr}Q^2$ is replaced by $\mathrm{Tr}\langle Q\rangle^2$.

We assume that the off-diagonal elements are set to their mean field values of zero, and consider only those components which will become nonzero in the uniaxial and biaxial phases.

We find the U-B transition curve given by, (as in previous section),

$$u_3 = \pm 12u_6\left(\frac{-r}{48(u_4 + w)}\right)^{3/2} = 12u_6M^3$$

with

$$\varphi_1 = \pm\left(\frac{-r}{48(u_4 + w)}\right)^{1/2} = M$$

and

$$\varphi_2 = 0$$

Note that w is positive, this means that the magnitude of M decreases, and so the steepness of the uniaxial-to-biaxial transition curve decreases as well.

6. RECURSION RELATIONS

Since we are looking at the second order uniaxial-to-biaxial phase transition it is natural to redefine our variables φ_1 first before proceeding with the renormalization group transformation.

We begin by rewriting the order parameter to reflect non-zero expectation value of φ_1 in the same way as we did for the incompressible case,

$$Q_{ij} = S_i \delta_{ij} + \hat{Q}_{ij}$$

where

$$S_1 = M + \varphi_1 + \varphi_2$$

$$S_2 = M + \varphi_1 - \varphi_2$$

$$S_3 = -2M - 2\varphi_1$$

$$\hat{Q}_{ij} = \begin{pmatrix} 0 & \hat{Q}_{12} & \hat{Q}_{13} \\ \hat{Q}_{12} & 0 & \hat{Q}_{23} \\ \hat{Q}_{13} & \hat{Q}_{23} & 0 \end{pmatrix}$$

As shown, the Q_{ij} is explicitly symmetric and traceless. In this form we can easily substitute into the equation for the effective Hamiltonian and compute the coefficients.

Having rewritten the effective Hamiltonian H_{eff} in terms of our new components of Q, (φ_1 renamed to $M + \varphi_1$), we keep terms up to 6th order in the elements of the tensor Q. Substituting the MFT values for M, as well as the coexistence curve equation $u_3 = 12u_6M^3$, we find that on the U-B transition curve, $r_1 \neq 0$, but $r_2 = r_3 = r_4 = r_5 = h = 0$. As before, this implies that φ_1 will undergo Gaussian fluctuations and so we will only need to keep terms which are quadratic in φ_1, but we must keep terms which are of fourth order in φ_2 and the off-diagonal elements for stability.

After dropping terms which are small, (the terms of order 5th and 6th will be irrelevant in the renormalization group sense), we get,

$$H = H_c + H_0 + H_1 + H_2 + H_3 + H_4 \tag{60}$$

$$H_c = L^d \left[\frac{3}{2} rM^2 - 6u_3M^3 + 36(u_4 + w)M^4 + 36u_6M^6\right] \tag{61}$$

$$H_0 = \int d^dx \left[\frac{3}{2} r_1\varphi_1^2 + \frac{1}{2} r_2\varphi_2^2 + \frac{1}{2} r_3\hat{Q}_{12}^2 + \frac{1}{2} r_4\hat{Q}_{13}^2 + \frac{1}{2} r_5\hat{Q}_{23}^2 + \frac{1}{4} \nabla_k Q_{ij}\nabla_k Q_{ij}\right] \tag{62}$$

$$H_1 = h \int d^dx\varphi_1(x) \tag{63}$$

$$H_2 = x \int d^dx\varphi_1(x)L^{-d} \int d^dy\varphi_1(y) \tag{64}$$

$$\begin{aligned} H_3 = \int d^dx \Big[&\varphi_1(y_1\varphi_2^2 + y_2\hat{Q}_{12}^2 + y_3\hat{Q}_{13}^2 + y_4\hat{Q}_{23}^2) \\ &+ y_5\varphi_2\hat{Q}_{13}^2 + y_6\varphi_2\hat{Q}_{23}^2 + y_7\hat{Q}_{12}\hat{Q}_{13}\hat{Q}_{23} \\ &+ y_8\varphi_1(x)L^{-d} \int d^dy\varphi_2^2(y) + y_9\varphi_1(x)L^{-d} \int d^dy\hat{Q}_{12}^2(y) \\ &+ y_{10}\varphi_1(x)L^{-d} \int d^dy\hat{Q}_{13}^2(y) \\ &+ y_{11}\varphi_1(x)L^{-d} \int d^dy\hat{Q}_{23}^2(y)\Big] \end{aligned} \tag{65}$$

$$\begin{aligned} H_4 = \int d^dx \Big[&z_1\varphi_2^4 + z_2\varphi_2^2\hat{Q}_{12}^2 + z_3\varphi_2^2\hat{Q}_{13}^2 + z_4\varphi_2^2\hat{Q}_{23}^2 \\ &+ z_5\hat{Q}_{12}^4 + z_6\hat{Q}_{13}^4 + z_7\hat{Q}_{23}^4 \\ &+ z_8\hat{Q}_{12}^2\hat{Q}_{13}^2 + z_9\hat{Q}_{12}^2\hat{Q}_{23}^2 + z_{10}\hat{Q}_{13}^2\hat{Q}_{23}^2 \\ &+ z_{11}\varphi_2^2L^{-d} \int d^dy\varphi_2^2 + z_{12}\varphi_2^2L^{-d} \int d^dy\hat{Q}_{12}^2(y) \\ &+ z_{13}\varphi_2^2L^{-d} \int d^dy\hat{Q}_{13}^2(y) + z_{14}\varphi_2^2L^{-d} \int d^dy\hat{Q}_{23}^2(y) \\ &+ z_{15}\hat{Q}_{12}^2(x)L^{-d} \int d^dy\hat{Q}_{12}^2(y) \\ &+ z_{16}\hat{Q}_{13}^2(x)L^{-d} \int d^dy\hat{Q}_{13}^2(y) \end{aligned}$$

$$+ \; z_{17}\hat{Q}^2_{23}(x)L-d \int d^dy\hat{Q}^2_{23}(y)$$

$$+ \; z_{18}\hat{Q}^2_{12}(x)L^{-d} \int d^dy\hat{Q}^2_{13}(y)$$

$$+ \; z_{19}\hat{Q}^2_{12}(x)L^{-d} \int d^dy\hat{Q}^2_{23}(y)$$

$$+ \; z_{20}\hat{Q}^2_{13}(x)L^{-d} \int d^dy\hat{Q}^2_{23}(y)\Bigg] \tag{66}$$

The quadratic coefficients are given by the equations,

$$r_1 = r - 12u_3M + 144u_4M^2 + 48wM^2 + 360u_6M^4 \tag{67}$$

$$r_2 = r + 12u_3M + 48(u_4 + w)M^2 - 144u_6M^4 \tag{68}$$

$$r_4 = r - 6u_3M + 48(u_4 + w)M^2 + 72u_6M^4 \tag{69}$$

and by $r_2 = r_3$, $r_4 = r_5$. The above equations are of the same form as in the incompressible case, with u_4 replaced by $u_4 + w$. We note that r_2, r_3, r_4, and r_5 vanish on the uniaxial-biaxial transition.

h is a coefficient to a linear term,

$$h = 3rM - 18u_3M^2 + 144(u_4 + w)M^3 + 216u_6M^5 \tag{70}$$

and h also vanishes on the U-B transition curve.

The coefficient in H_2 is,

$$x = 144wM^2 \tag{71}$$

The third order coefficients are,

$$y_1 = 6u_3 + 48u_4M - 288u_6M^3 \tag{72}$$

$$y_3 = -3u_3 + 48u_4M + 144u_6M^3 \tag{73}$$

$$y_5 = 3u_3 - 36u_6M^3 \tag{74}$$

$$y_8 = 48wM \tag{75}$$

where $y_1 = y_2$, $y_3 = y_4$, $y_5 = -y_6 = \frac{1}{2}y_7$, and $y_8 = y_9 = y_{10} = y_{11}$.

The fourth order coefficients are,

$$z_1 = 4u_4 + 36u_6M^2 \tag{76}$$

$$z_3 = 8u_4 - 36u_6M^2 \tag{77}$$

$$z_6 = 4u_4 + 9u_6M^2 \tag{78}$$

$$z_{11} = 4w \tag{79}$$

$$z_{13} = 8w \tag{80}$$

$$z_{16} = 4w \tag{81}$$

The rest of the fourth order coefficients are given by the relations, $z_1 = \frac{1}{2}z_2 = z_5$, $z_3 = z_4 = z_8 = z_9$, $z_6 = z_7 = \frac{1}{2}z_{10}$, $z_{11} = \frac{1}{2}z_{12} = z_{15}$, $z_{13} = z_{14} = z_{18} = z_{19}$, and $z_{16} = z_{17} = \frac{1}{2}z_{20}$.

The φ_1 variables can be integrated immediately. We integrate after completing the square,

$$\begin{aligned}\exp[-H_{\text{eff}}] = \exp[-(H_0' + H_3' + H_4)] \\ \times \int \mathcal{D}\varphi_1 \exp\left[-\frac{3}{2}L^{-d}\sum_k\left(r_1 + k^2 + \frac{2}{3}x\delta(k)\right)\right. \\ \left.\varphi_1(k)\varphi_1(-k) - \frac{3}{2}L^{-d}\sum_k \frac{2}{3}\varphi_1(k)g(-k)\right]\end{aligned} \tag{82}$$

$$\begin{aligned}= \exp[-(H_0' + H_3' + H_4)] \\ \times \int \mathcal{D}\varphi_1 \exp\left[-\frac{3}{2}L^{-d}\sum_k\left(r_1 + k_2 + \frac{2}{3}x\delta(k)\right)\right. \\ \times\left(\varphi_1(k) + \frac{1}{3}\frac{g(k)}{r_1 + k^2 + \frac{2}{3}x\delta(k)}\right) \\ \left.\times\left(\varphi_1(-k) + \frac{1}{3}\frac{g(-k)}{r_1 + k^2 + \frac{2}{3}x\delta(k)}\right)\right] \\ \times \exp\left[\frac{1}{6}L^{-d}\sum_k \frac{1}{r_1 + k^2 + \frac{2}{3}x\delta(k)}g(k)g(-k)\right]\end{aligned} \tag{83}$$

Where we have set,

$$H_0' = \frac{1}{2} L^{-d} \sum_{\mathbf{k}} (r_2 + k^2)\varphi_2(\mathbf{k})\varphi_2(-\mathbf{k})$$

$$+ \frac{1}{2} L^{-d} \sum_{\mathbf{k}} (r_3 + k^2)\hat{Q}_{12}(\mathbf{k})\hat{Q}_{12}(-\mathbf{k})$$

$$+ \frac{1}{2} L^{-d} \sum_{\mathbf{k}} (r_4 + k^2)\hat{Q}_{13}(\mathbf{k})\hat{Q}_{13}(-\mathbf{k})$$

$$+ \frac{1}{2} L^{-d} \sum_{\mathbf{k}} (r_5 + k^2)\hat{Q}_{23}(\mathbf{k})\hat{Q}_{23}(-\mathbf{k})$$

$$H_3' = L^{-2d} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)[y_5\varphi_2(\mathbf{k}_1)\hat{Q}_{13}(\mathbf{k}_2)\hat{Q}_{13}(\mathbf{k}_3)$$

$$+ y_6\varphi_2(\mathbf{k}_1)\hat{Q}_{23}(\mathbf{k}_2)\hat{Q}_{23}(\mathbf{k}_3) + y_7\hat{Q}_{12}(\mathbf{k}_1)\hat{Q}_{13}(\mathbf{k}_2)\hat{Q}_{23}(\mathbf{k}_3)]$$

$$g(-\mathbf{q}) = \delta(-\mathbf{q})L^d \Big[h + \sum_{\mathbf{k}} (y_8\varphi_2(\mathbf{k})\varphi_2(-\mathbf{k}) + y_9\hat{Q}_{12}(\mathbf{k})\hat{Q}_{12}(-\mathbf{k})$$

$$+ y_{10}\hat{Q}_{13}(\mathbf{k})\hat{Q}_{13}(-\mathbf{k}) + y_{11}\hat{Q}_{23}(\mathbf{k})\hat{Q}_{23}(-\mathbf{k})) \Big]$$

$$+ L^{-d} \sum_{\mathbf{k}_1,\mathbf{k}_2} \delta(\mathbf{q} + \mathbf{k}_1 + \mathbf{k}_2)[y_1\varphi_2(\mathbf{k}_1)\varphi_2(\mathbf{k}_2)$$

$$+ y_2\hat{Q}_{12}(\mathbf{k}_1)\hat{Q}_{12}(\mathbf{k}_2) + y_3\hat{Q}_{13}(\mathbf{k}_1)\hat{Q}_{13}(\mathbf{k}_2) + y_4\hat{Q}_{23}(\mathbf{k}_1)\hat{Q}_{23}(\mathbf{k}_2)]$$

After expanding and multiplying the g's, we collect like terms and end up with the effective Hamiltonian,

$$H_{\text{eff}} = \frac{1}{2} L^{-d} \sum_{k} (\tau_2 + k^2)\varphi_2(k)\varphi_2(-k)$$

$$+ \frac{1}{2} L^{-d} \sum_{k} (\tau_3 + k^2)\hat{Q}_{12}(k)\hat{Q}_{12}(-k)$$

$$+ \frac{1}{2} L^{-d} \sum_{k} (\tau_4 + k^2)\hat{Q}_{13}(k)\hat{Q}_{13}(-k)$$

$$+ \frac{1}{2} L^{-d} \sum_{k} (\tau_5 + k^2)\hat{Q}_{23}(k)\hat{Q}_{23}(-k)$$

$$+ L^{-2d} \sum_{k_1,k_2,k_3} \delta(k_1 + k_2 + k_3)[y_5\varphi_2(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{13}(k_3)$$

$$+ y_6\varphi_2(k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(k_3) + y_7\hat{Q}_{12}(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{23}(k_3)]$$

$$+ L^{-3d} \sum_{k_1,k_2,k_3,k_4} \delta(k_1 + k_2 + k_3 + k_4)$$

$$\times [v_1(k_1 + k_2)\varphi_2(k_1)\varphi_2(k_2)\varphi_2(k_3)\varphi_2(k_4)$$

$$+ v_2(k_1 + k_2)\varphi_2(k_1)\varphi_2(k_2)\hat{Q}_{12}(k_3)\hat{Q}_{12}(k_4)$$

$$+ v_3(k_1 + k_2)\varphi_2(k_1)\varphi_2(k_2)\hat{Q}_{13}(k_3)\hat{Q}_{13}(k_4)$$

$$+ v_4(k_1 + k_2)\varphi_2(k_1)\varphi_2(k_2)\hat{Q}_{23}(k_3)\hat{Q}_{23}(k_4)$$

$$+ v_5(k_1 + k_2)\hat{Q}_{12}(k_1)\hat{Q}_{12}(k_2)\hat{Q}_{12}(k_3)\hat{Q}_{12}(k_4)$$

$$+ v_6(k_1 + k_2)\hat{Q}_{13}(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{13}(k_3)\hat{Q}_{13}(k_4)$$

$$+ v_7(k_1 + k_2)\hat{Q}_{23}(k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(k_3)\hat{Q}_{23}(k_4)$$

$$+ v_8(k_1 + k_2)\hat{Q}_{12}(k_1)\hat{Q}_{12}(k_2)\hat{Q}_{13}(k_3)\hat{Q}_{13}(k_4)$$

$$+ v_9(k_1 + k_2)\hat{Q}_{12}(k_1)\hat{Q}_{12}(k_2)\hat{Q}_{23}(k_3)\hat{Q}_{23}(k_4)$$

$$+ v_{10}(k_1 + k_2)\hat{Q}_{13}(k_1)\hat{Q}_{13}(k_2)\hat{Q}_{23}(k_3)\hat{Q}_{23}(k_4)]$$

$$+ L^{-3d} \sum_{\mathbf{k}_1,\mathbf{k}_2} [v_{11}\varphi_2(k_1)\varphi_2(-k_1)\varphi_2(k_2)\varphi_2(-k_2)$$

$$+ v_{12}\varphi_2(k_1)\varphi_2(-k_1)\hat{Q}_{12}(k_2)\hat{Q}_{12}(-k_2)$$

$$+ v_{13}\varphi_2(k_1)\varphi_2(-k_1)\hat{Q}_{13}(k_2)\hat{Q}_{13}(-k_2)$$

$$+ v_{14}\varphi_2(k_1)\varphi_2(-k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(-k_2)$$

$$+ v_{15}\hat{Q}_{12}(k_1)\hat{Q}_{12}(-k_1)\hat{Q}_{12}(k_2)\hat{Q}_{12}(-k_2)$$

$$+ v_{16}\hat{Q}_{13}(k_1)\hat{Q}_{13}(-k_1)\hat{Q}_{13}(k_2)\hat{Q}_{13}(-k_2)$$

$$+ v_{17}\hat{Q}_{23}(k_1)\hat{Q}_{23}(-k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(-k_2)$$

$$+ v_{18}\hat{Q}_{12}(k_1)\hat{Q}_{12}(-k_1)\hat{Q}_{13}(k_2)\hat{Q}_{13}(-k_2)$$

$$+ v_{19}\hat{Q}_{12}(k_1)\hat{Q}_{12}(-k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(-k_2)$$

$$+ v_{20}\hat{Q}_{13}(k_1)\hat{Q}_{13}(-k_1)\hat{Q}_{23}(k_2)\hat{Q}_{23}(-k_2)]$$

The next step is to use the momentum shell technique.[24] We first integrate away the degrees of freedom with wave vectors between $b^{-1} < |\mathbf{k}| < 1$, with $b > 1$. Then convert these non-linear equations into algebraic equations as in Wilson and Kogut,[11] the momentum dependence is of $O(\varepsilon)$ and is small compared to r_1. We justify this by choosing a point long the U-B transition curve ($u_3 = 12u_6M^3$), such that $-r$ is of $O(1)$, and y_1^2 is of $O(\varepsilon)$ (see Reference 29). The recursion relations are illustrated graphically in Figure B.5.

We can compute the differential recursion relations from the recursion relations by letting $b = e^{\delta l}$, δl infinitesimal. We find that the same relations among the coefficients that hold in MFT also hold under the differential recursion equations, so we only need to consider 2 of the 4 quadratic coefficients, 1 of the 3 third order coefficients, and 6 of the 20 quartic coefficients.

The final set of recursion relations are,

$$\tau_2' = b^2(\tau_2 + K_4[8v_1(1 - b^{-2} - 2\tau_2 \ln b) + 2v_3(1 - b^2 - 2\tau_4 \ln b)$$
$$+ 4v_{11}(1 - b^{-2} - 2\tau_2 \ln b) + 2v_{13}(1 - b^{-2} - 2\tau_4 \ln b) - 4y_5^2 \ln b]) \quad (84)$$

$$\tau_4' = b^2(\tau_4 + K_4[2v_3(1 - b^{-2} - 2\tau_2 \ln b) + 8v_6(1 - b^{-2} - 2\tau_4 \ln b)$$
$$+ 2v_{13}(1 - b^{-2} - 2\tau_2 \ln b) + 4v_{16}(1 - b^{-2} - 2\tau_4 \ln b) - 8y_5^2 \ln b]) \quad (85)$$

$$y_5' = b^{1+\varepsilon/2}(y_5 - K_4[8v_3y_5 \ln b + 8v_6y_5 \ln b]) \quad (86)$$

$$v_1' = b^\varepsilon(v_1 - K_4[40v_1^2 \ln b + 2v_3^2 \ln b) + K_4[4v_3y_5^2(b^2 - 1) - y_5^4(b^4 - 1)]) \quad (87)$$

$$v_3' = b^\varepsilon(v_3 - K_4[8v_3^2 \ln b + 16(v_1 + v_6)v_3 \ln b]$$
$$+ K_4[16v_1y_5^2 + 12\, v_3y_5^2 + 16v_6y_5^2](b^2 - 1) - 4K_4y_5^4(b^4 - 1)) \quad (88)$$

$$v_6' = b^\varepsilon(v_6 - K_4[40v_6^2 \ln b + 2v_3^2 \ln b]$$
$$+ K_4[4v_3y_5^2 + 16v_6y_5^2](b^2 - 1) - 2K_4y_5^4(b^4 - 1)) \quad (89)$$

$$v_{11}' = b^\varepsilon(v_{11} - K_4 \ln b[8v_{11}^2 + 2v_{13}^2 + 32v_1v_{11} + 4v_3v_{13}] + 4K_4v_{13}y_5^2(b^2 - 1)) \quad (90)$$

$$v_{13}' = b^\varepsilon(v_{13} - K_4 \ln b[16v_{13}(v_1 + v_6) + 8v_{13}(v_{11} + v_{16})$$
$$+ 8v_3(v_{11} + v_{16})] + K_4[8v_{11}y_5^2 + 4v_{13}y_5^2 + 8v_{16}y_5^2](b^2 - 1)) \quad (91)$$

$$v_{16}' = b^\varepsilon(v_{16} - K_4 \ln b\, [8v_{16}^2 + 2v_{13}^2 + 32v_6v_{16} + 4v_3v_{13}]$$
$$+ K_4[4v_{13}y_5^2 + 8v_{16}y_5^2](b^2 - 1)) \quad (92)$$

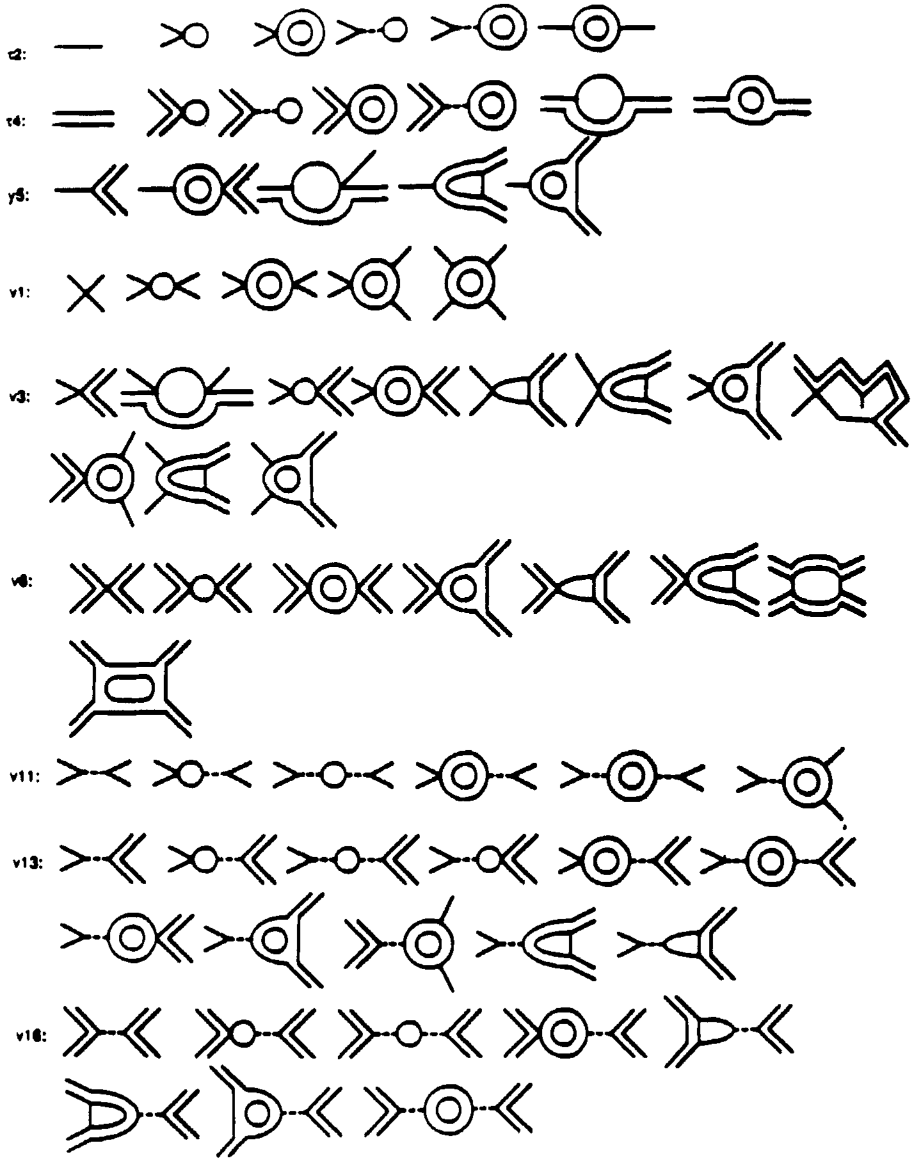

FIGURE B.5 Diagrammatic representation of the recursion relations for compressible liquid crystals. The single line represents φ_2, while the double lines represents either Q_{12}, Q_{13}, or Q_{23}.

where $K_4 = 1/8\pi^2$.

Since there are 6 quartic coefficients we consider the $2^6 = 64$ possible combinations of zero and non-zero v's. These yield 18 fixed points and 2 continuous fixed points. We linearize these equations about the fixed points to compute the respective critical exponents by letting, for instance, $v_1' = v_1^* + \delta v_1'$. The linearized

TABLE B.1

Fixed points and λ-exponents for the incompressible liquid crystal at the U-B transition to $O(\varepsilon)$.

F	τ_2	τ_4	y_5	v_1	v_3	v_6	$\lambda_{\tau 2}$	$\lambda_{\tau 4}$	λ_{y5}	λ_{v1}	λ_{v3}	λ_{v6}
1	0	0	0	0	0	0	2	2	$1+\frac{1}{2}\epsilon$	ϵ	ϵ	ϵ
2	$-\frac{1}{5}\epsilon$	0	0	$\frac{2\pi^2}{10}\epsilon$	0	0	$2-\frac{2}{5}\epsilon$	2	$1+\frac{1}{2}\epsilon$	$-\epsilon$	$\frac{3}{5}\epsilon$	ϵ
3	0	$-\frac{\epsilon}{5}$	0	0	0	$\frac{2\pi^2}{10}\epsilon$	2	$2-\frac{2}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	ϵ	$\frac{3}{5}\epsilon$	$-\epsilon$
4	$-\frac{\epsilon}{5}$	$-\frac{\epsilon}{5}$	0	$\frac{2\pi^2}{10}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	$2-\frac{2}{5}\epsilon$	$2-\frac{2}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	$-\epsilon$	$\frac{1}{5}\epsilon$	$-\epsilon$

renormalization group matrix is upper-triangular at fixed points F1 through F16, see Table B.1), when $v_3^* = 0$, $v_{13}^* = 0$, and the basis vector is rearranged,

$$(\tau_2, \tau_4, y_5, v_{11}, v_{13}, v_{16}, v_1, v_3, v_6)$$

The exponents for fixed points F1 through F16 can be written as,

$$\lambda_{\tau 2} = 2 - 8K_4(2v_1^* + v_{11}^*) \tag{93}$$

$$\lambda_{\tau 4} = 2 - 8K_4(2v_6^* + v_{16}^*) \tag{94}$$

$$\lambda_{y5} = 1 + \frac{\varepsilon}{2} - 8K_4(v_3^* + v_6^*) \tag{95}$$

$$\lambda_{v1} = \varepsilon - 80K_4v_1^* \tag{96}$$

$$\lambda_{v3} = \varepsilon - 16K_4(v_1^* + v_3^* + v_6^*) \tag{97}$$

$$\lambda_{v6} = \varepsilon - 80K_4v_6^* \tag{98}$$

$$\lambda_{v11} = \varepsilon - 16K_4(2v_1^* + v_{11}^*) \tag{99}$$

$$\lambda_{v13} = \varepsilon - 8K_4(2v_1^* + 2v_6^* + v_{11}^* + v_{16}^*) \tag{100}$$

$$\lambda_{v16} = \varepsilon - 16K_4(2v_6^* + v_{16}^*) \tag{101}$$

The critical exponents for fixed point F17 are,

$$\lambda_{1,2} = 2 - \frac{\varepsilon}{3} \mp \frac{\varepsilon}{6} \tag{102}$$

$$\lambda_{y5} = 1 \tag{103}$$

$$\lambda_3 = \frac{\varepsilon}{3} \tag{104}$$

$$\lambda_4 = -\frac{2\varepsilon}{3} \tag{105}$$

$$\lambda_{5,6} = \frac{\varepsilon}{3} \pm \frac{\varepsilon}{3} \tag{106}$$

$$\lambda_{7,8} = -\frac{\varepsilon}{2} \pm \frac{\varepsilon}{2} \tag{107}$$

while the exponents for F18 are,

$$\lambda_{1,2} = 2 - \frac{2\varepsilon}{3} \pm \frac{\varepsilon}{6} \tag{108}$$

$$\lambda_{y5} = 1 \tag{109}$$

$$\lambda_3 = -\frac{\varepsilon}{3} \tag{110}$$

$$\lambda_4 = -\frac{2\varepsilon}{3} \tag{111}$$

$$\lambda_{5,6} = -\frac{\varepsilon}{3} \pm \frac{\varepsilon}{3} \tag{112}$$

$$\lambda_{7,8} = -\frac{\varepsilon}{2} \pm \frac{\varepsilon}{2} \tag{113}$$

We may treat the infinite range energy density coupling coefficients, v_{11}, v_{13}, and v_{16} by a mean field theory assumption, as in Sak.[23] We can use a simple counting of dimensions where we use the fact that the energy-density has dimension $\frac{1}{\nu}(1 - \alpha)$ and $d\nu = 2 - \alpha$, (the hyperscaling law), to find a relation among $\lambda_{v_{11}}$, $\lambda_{v_{13}}$, and $\lambda_{v_{16}}$. When $v_{13} = 0$ we have a decoupled set of exponents, and we find (using arguments similar to Aharony[17] and Wegner[30]),

$$\lambda_{v13} = \frac{1}{2}\left(\frac{\alpha_1}{\nu_1} + \frac{\alpha_2}{\nu_2}\right)$$

the exponents are given as,

$$\lambda_{v11} = \frac{\alpha_1}{\nu_1} \tag{114}$$

$$\lambda_{v16} = \frac{\alpha_2}{\nu_2} \tag{115}$$

The 2 continuous fixed points arise because of accidental cancellations which reduce the number of independent equations. The first continuous fixed point occurs when $v_1 = v_3 = v_6 = 0$ and v_{11}, v_{13} and v_{16} are non-zero. Setting $dv_{11}/dl = dv_{11}/dl = dv_{16}/dl = 0$ we get,

$$\varepsilon v_{11} - K_4(8v_{11}^2 + 2v_{13}^2) = 0 \tag{116}$$

$$\varepsilon v_{13} - 8K_4(v_{11} + v_{16})v_{13} = 0 \tag{117}$$

$$\varepsilon v_{16} - K_4(8v_{16}^2 + 2v_{13}^2) = 0 \tag{118}$$

Since v_{13} is assumed to be non-zero, we may divide out v_{13} from the second equation which leaves us with,

$$v_{11} + v_{16} = \frac{\varepsilon}{8K_4} \tag{119}$$

If we subtract the first equation from the third we get,

$$\varepsilon(v_{11} - v_{16}) - 8K_4(v_{11} - v_{16})(v_{11} + v_{16}) = 0 \tag{120}$$

If we substitute $v_{11} + v_{16} = \varepsilon/8K_4$ into the above, we see that this system of equations is dependent. Solving for v_{13} in terms of v_{11},

$$\varepsilon v_{11} - K_4(8v_{11}^2 + 2v_{13}^2) = 0 \tag{121}$$

$$v_{13} = \pm 2\left[\frac{\varepsilon}{8K_4} v_{11} - v_{11}^2\right]^{1/2} \tag{122}$$

Since v_{13} must be real, we see that the domain of v_{11} is restricted to $0 < v_{11} < \varepsilon/8K_4$, and similarly $v_{11} + v_{16} = \varepsilon/8K_4$ implies that v_{16} is limited to $0 < v_{16} < \varepsilon/8K_4$.

The second continuous fixed point occurs when $v_1 = v_6 = \varepsilon/40K_4$, $v_3 = 0$ and v_{11}, v_{13} and v_{16} are non-zero. Setting $dv_{11}/dl = dv_{13}/dl = dv_{16}/dl = 0$ we get,

$$\frac{\varepsilon}{5} v_{11} - K_4(8v_{11}^2 + 2v_{13}^2) = 0 \tag{123}$$

$$\frac{\varepsilon}{5} v_{13} - 8K_4 v_{13}(v_{11} + v_{16}) = 0 \tag{124}$$

$$\frac{\varepsilon}{5} v_{16} - K_4(8v_{16}^2 + 2v_{13}^2) = 0 \tag{125}$$

In the same way, we can solve for v_{13} in terms of v_{11},

$$v_{13} = \pm 2 \left[\frac{\varepsilon}{40K_4} v_{11} - v_{11}^2 \right]^{1/2} \tag{126}$$

v_{13} is real, so the domain of v_{11} is $0 < v_{11} < \varepsilon/40K_4$, and $v_{11} + v_{16} = \varepsilon/40K_4$ implies that v_{16} is limited to $0 < v_{16} < \varepsilon/40K_4$.

The linearized RG matrix gives a complicated set of critical exponents, however, these fixed points are unstable in at least one critical exponent so we don't expect the system will manifest this type of behavior.

Considering the 18 fixed points, fixed points F1–F17 are unstable in at least one coefficient, so the critical behavior is not expected to be represented by these fixed points. F18 is the most stable fixed point, and it is accessible since v_{11}^* and v_{16}^* are both positive in the mean field approximation. As before, $1/\nu = 2 - \varepsilon/2$, and $\alpha = 0 + O(\varepsilon^2)$, which is consistent with the interpretation that the number of critical degrees of freedom is 4. This leads us to conclude that the system undergoes a second order transition, where the critical behavior is given by the fixed point F18. So the addition of compressibility into the system does not change the order of the transition as we expected since the critical heat exponent α was not positive for the incompressible case.

TABLE B.2

Fixed points for the compressible liquid crystal at the U-B transition to $O(\varepsilon)$.

F	r_2	r_4	y_5	v_1	v_3	v_6	v_{11}	v_{13}	v_{16}
1	0	0	0	0	0	0	0	0	0
2	0	$-\frac{1}{2}\epsilon$	0	0	0	0	0	0	$\frac{2\pi^2}{2}\epsilon$
3	$-\frac{1}{2}\epsilon$	0	0	0	0	0	$\frac{2\pi^2}{2}\epsilon$	0	0
4	$-\frac{1}{2}\epsilon$	$-\frac{1}{2}\epsilon$	0	0	0	0	$\frac{2\pi^2}{2}\epsilon$	0	$\frac{2\pi^2}{2}\epsilon$
5	0	$-\frac{1}{5}\epsilon$	0	0	0	$\frac{2\pi^2}{10}\epsilon$	0	0	0
6	0	$-\frac{3}{10}\epsilon$	0	0	0	$\frac{2\pi^2}{10}\epsilon$	0	0	$\frac{2\pi^2}{10}\epsilon$
7	$-\frac{1}{2}\epsilon$	$-\frac{1}{5}\epsilon$	0	0	0	$\frac{2\pi^2}{10}\epsilon$	$\frac{2\pi^2}{2}\epsilon$	0	0
8	$-\frac{1}{2}\epsilon$	$-\frac{3}{10}\epsilon$	0	0	0	$\frac{2\pi^2}{10}\epsilon$	$\frac{2\pi^2}{2}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$
9	$-\frac{1}{5}\epsilon$	0	0	$\frac{2\pi^2}{10}\epsilon$	0	0	0	0	0
10	$-\frac{1}{5}\epsilon$	$-\frac{1}{2}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	0	0	0	0	$\frac{2\pi^2}{2}\epsilon$
11	$-\frac{3}{10}\epsilon$	0	0	$\frac{2\pi^2}{10}\epsilon$	0	0	$\frac{2\pi^2}{10}\epsilon$	0	0
12	$-\frac{3}{10}\epsilon$	$-\frac{1}{2}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	0	0	$\frac{2\pi^2}{10}\epsilon$	0	$\frac{2\pi^2}{2}\epsilon$
13	$-\frac{1}{5}\epsilon$	$-\frac{1}{5}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	0	0	0
14	$-\frac{1}{5}\epsilon$	$-\frac{3}{10}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	0	0	$\frac{2\pi^2}{10}\epsilon$
15	$-\frac{3}{10}\epsilon$	$-\frac{1}{5}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	$\frac{2\pi^2}{10}\epsilon$	0	0
16	$-\frac{3}{10}\epsilon$	$-\frac{3}{10}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$	$\frac{2\pi^2}{10}\epsilon$	0	$\frac{2\pi^2}{10}\epsilon$
17	$-\frac{1}{4}\epsilon$	$-\frac{1}{4}\epsilon$	0	$\frac{2\pi^2}{12}\epsilon$	$\frac{2\pi^2}{6}\epsilon$	$\frac{2\pi^2}{12}\epsilon$	0	0	0
18	$-\frac{1}{4}\epsilon$	$-\frac{1}{4}\epsilon$	0	$\frac{2\pi^2}{12}\epsilon$	$\frac{2\pi^2}{6}\epsilon$	$\frac{2\pi^2}{12}\epsilon$	$\frac{2\pi^2}{6}\epsilon$	$-\frac{2\pi^2}{3}\epsilon$	$\frac{2\pi^2}{6}\epsilon$

TABLE B.3

λ-Exponents for the compressible liquid crystal at the U-B transition to $O(\varepsilon)$.

F	λ_{r2}	λ_{r4}	λ_{y5}	λ_{v1}	λ_{v3}	λ_{v6}	λ_{v11}	λ_{v13}	λ_{v16}
1	2	2	$1+\frac{1}{2}\epsilon$	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
2	2	$2-\epsilon$	$1+\frac{1}{2}\epsilon$	ϵ	ϵ	ϵ	ϵ	0	$-\epsilon$
3	$2-\epsilon$	2	$1+\frac{1}{2}\epsilon$	ϵ	ϵ	ϵ	$-\epsilon$	0	ϵ
4	$2-\epsilon$	$2-\epsilon$	$1+\frac{1}{2}\epsilon$	ϵ	ϵ	ϵ	$-\epsilon$	$-\epsilon$	$-\epsilon$
5	2	$2-\frac{2}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	ϵ	$\frac{3}{5}\epsilon$	$-\epsilon$	ϵ	$\frac{3}{5}\epsilon$	$\frac{1}{5}\epsilon$
6	2	$2-\frac{3}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	ϵ	$\frac{3}{5}\epsilon$	$-\epsilon$	ϵ	$\frac{2}{5}\epsilon$	$-\frac{1}{5}\epsilon$
7	$2-\epsilon$	$2-\frac{2}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	ϵ	$\frac{3}{5}\epsilon$	$-\epsilon$	$-\epsilon$	$-\frac{2}{5}\epsilon$	$\frac{1}{5}\epsilon$
8	$2-\epsilon$	$2-\frac{3}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	ϵ	$\frac{3}{5}\epsilon$	$-\epsilon$	$-\epsilon$	$-\frac{3}{5}\epsilon$	$-\frac{1}{5}\epsilon$
9	$2-\frac{2}{5}\epsilon$	2	$1+\frac{1}{2}\epsilon$	$-\epsilon$	$\frac{3}{5}\epsilon$	ϵ	$\frac{1}{5}\epsilon$	$\frac{3}{5}\epsilon$	ϵ
10	$2-\frac{2}{5}\epsilon$	$2-\epsilon$	$1+\frac{1}{2}\epsilon$	$-\epsilon$	$\frac{3}{5}\epsilon$	ϵ	$\frac{1}{5}\epsilon$	$-\frac{2}{5}\epsilon$	$-\epsilon$
11	$2-\frac{3}{5}\epsilon$	2	$1+\frac{1}{2}\epsilon$	$-\epsilon$	$\frac{3}{5}\epsilon$	ϵ	$-\frac{1}{5}\epsilon$	$\frac{2}{5}\epsilon$	ϵ
12	$2-\frac{3}{5}\epsilon$	$2-\epsilon$	$1+\frac{1}{2}\epsilon$	$-\epsilon$	$\frac{3}{5}\epsilon$	ϵ	$-\frac{1}{5}\epsilon$	$-\frac{3}{5}\epsilon$	$-\epsilon$
13	$2-\frac{2}{5}\epsilon$	$2-\frac{2}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	$-\epsilon$	$\frac{1}{5}\epsilon$	$-\epsilon$	$\frac{1}{5}\epsilon$	$\frac{1}{5}\epsilon$	$\frac{1}{5}\epsilon$
14	$2-\frac{2}{5}\epsilon$	$2-\frac{3}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	$-\epsilon$	$\frac{1}{5}\epsilon$	$-\epsilon$	$\frac{1}{5}\epsilon$	0	$-\frac{1}{5}\epsilon$
15	$2-\frac{3}{5}\epsilon$	$2-\frac{2}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	$-\epsilon$	$\frac{1}{5}\epsilon$	$-\epsilon$	$-\frac{1}{5}\epsilon$	0	$\frac{1}{5}\epsilon$
16	$2-\frac{3}{5}\epsilon$	$2-\frac{3}{5}\epsilon$	$1+\frac{3}{10}\epsilon$	$-\epsilon$	$\frac{1}{5}\epsilon$	$-\epsilon$	$-\frac{1}{5}\epsilon$	$-\frac{1}{5}\epsilon$	$-\frac{1}{5}\epsilon$

In 1984, Boonbrahm and Saupe[31] studied the uniaxial to biaxial phase transition in an amphiphilic system, which is composed of molecules in an aqueous solution that are hydrophilic on one end and lipophilic on the other, leading to a clumping of the molecules into micelles. They find that the critical exponents are $\beta = 0.37 \pm 0.03$ and $\gamma = 1.33 \pm 0.04$, and interpret their results as being in agreement with the 3 dimensional XY-model. However, this is also consistent with our results, (which is an $O(\varepsilon)$ calculation), with $\beta = 0.375$ and $\gamma = 1.25$ when $\varepsilon = 1$. Since we have 4 fluctuating variables, we may also compare the experimental results to the critical exponent calculations for the $n = 4$ vector model which have been done to $O(\varepsilon^3)$, $\beta = 0.362$ and $\gamma = 1.37$ in 3-dimensions. Such comparisons cannot be taken too seriously since the ε expansion is no longer valid when $\varepsilon = 1$. Clearly, there is need for better accuracy both in the realm of theory and experiment to distinguish between the two models.

APPENDIX A

In the incompressible case, we integrate out φ_1 and the quadratic coefficients are relabeled as,

$$\tau_2 = r_2 - \frac{2hy_1}{3r_1} \tag{A1}$$

$$\tau_3 = r_3 - \frac{2hy_2}{3r_1} \tag{A2}$$

$$\tau_4 = r_4 - \frac{2hy_3}{3r_1} \tag{A3}$$

$$\tau_5 = r_5 - \frac{2hy_4}{3r_1} \tag{A4}$$

The quartic coefficients are,

$$v_1(q) = z_1 - \frac{1}{6}\frac{y_1^2}{r_1 + q^2} \tag{A5}$$

$$v_2(q) = z_2 - \frac{1}{3}\frac{y_1y_2}{r_1 + q^2} \tag{A6}$$

$$v_3(q) = z_3 - \frac{1}{3}\frac{y_1y_3}{r_1 + q^2} \tag{A7}$$

$$v_4(q) = z_4 - \frac{1}{3}\frac{y_1y_4}{r_1 + q^2} \tag{A8}$$

$$v_5(q) = z_5 - \frac{1}{6}\frac{y_2^2}{r_1 + q^2} \tag{A9}$$

$$v_6(q) = z_6 - \frac{1}{6}\frac{y_3^2}{r_1 + q^2} \tag{A10}$$

$$v_7(q) = z_7 - \frac{1}{6}\frac{y_4^2}{r_1 + q^2} \tag{A11}$$

$$v_8(q) = z_8 - \frac{1}{3}\frac{y_2y_3}{r_1 + q^2} \tag{A12}$$

$$v_9(q) = z_9 - \frac{1}{3}\frac{y_2y_4}{r_1 + q^2} \tag{A13}$$

$$v_{10}(q) = z_{10} - \frac{1}{3}\frac{y_3y_4}{r_1 + q^2} \tag{A14}$$

These new coefficients obey the mean field relations, $\tau_2 = \tau_3$, $\tau_4 = \tau_5$, $v_1 = \frac{1}{2}v_2 = v_5$, $v_3 = v_4 = v_8 = v_9$, $v_6 = v_7 = \frac{1}{2}v_{10}$, and will be useful in finding the fixed points.

The differential recursion relations for the quadratic coefficients are, (on an incompressible lattice),

$$\frac{d\tau_2}{dl} = 2\tau_2 + K_4(12v_1(1 - \tau_2) + 2v_2(1 - \tau_3) + 2v_3(1 - \tau_4)$$
$$+ 2v_4(1 - \tau_5) - 2y_5^2(1 - 2\tau_4) - 2y_6^2(1 - 2\tau_5)) \tag{A15}$$

$$\frac{d\tau_3}{dl} = 2\tau_3 + K_4(2v_2(1 - \tau_2) + 12v_5(1 - \tau_3) + 2v_8(1 - \tau_4) + 2v_9(1 - \tau_5) - 2y_7^2(1 - \tau_4 - \tau_5)) \quad \text{(A16)}$$

$$\frac{d\tau_4}{dl} = 2\tau_4 + K_4(2v_3(1 - \tau_2) + 2v_8(1 - \tau_3) + 12v_6(1 - \tau_4) + 2v_{10}(1 - \tau_5) - 4y_5^2(1 - \tau_2 - \tau_4) - y_7^2(1 - \tau_3 - \tau_5)) \quad \text{(A17)}$$

$$\frac{d\tau_5}{dl} = 2\tau_5 + K_4(2v_4(1 - \tau_2) + 2v_9(1 - \tau_3) + 2v_{10}(1 - \tau_4) + 12v_7(1 - \tau_5) - 4y_6^2(1 - \tau_2 - \tau_5) - y_7^2(1 - \tau_3 - \tau_4)) \quad \text{(A18)}$$

A quick check reveals $d\tau_2/dl = d\tau_3/dl$ and $d\tau_4/dl = d\tau_5/dl$ on the uniaxial-biaxial transition.

The third order coefficients are given by,

$$\frac{dy_5}{dl} = \left(1 + \frac{\varepsilon}{2}\right) y_5 - K_4(8v_3y_5 + 12v_6y_5 + 2v_{10}y_6 - 4y_5^3 - y_6y_7^2) \quad \text{(A19)}$$

$$\frac{dy_6}{dl} = \left(1 + \frac{\varepsilon}{2}\right) y_6 - K_4(8v_4y_6 + 12v_7y_6 + 2v_{10}y_5 - y_5y_7^2 - 4y_6^3) \quad \text{(A20)}$$

$$\frac{dy_7}{dl} = \left(1 + \frac{\varepsilon}{2}\right) y_7 - K_4y_7(4v_8 + 4v_9 + 4v_{10} - 4y_5y_6 - y_7^2) \quad \text{(A21)}$$

From these relations we see that on the uniaxial-biaxial transition curve,

$$\frac{dy_5}{dl} = -\frac{dy_6}{dl} = \frac{1}{2}\frac{dy_7}{dl}$$

The recursion relations for the fourth order coefficients are,

$$\frac{dv_1}{dl} = \varepsilon v_1 - K_4(36v_1^2 + v_2^2 + v_3^2 + v_4^2 - 4v_3y_5^2 - 4v_4y_6^2 + 2y_5^4 + 2y_6^4) \quad \text{(A22)}$$

$$\frac{dv_2}{dl} = \varepsilon v_2 - K_4(12v_1v_2 + 8v_2^2 + 12v_2v_5 + 2v_3v_8 + 2v_4v_9 - v_3y_7^2 - v_4y_7^2 - 4v_8y_5^2 - 4v_9y_6^2 + 2y_5^2y_7^2 + 2y_6^2y_7^2 + 2y_5y_6y_7^2) \quad \text{(A23)}$$

$$\frac{dv_3}{dl} = \varepsilon v_3 - K_4(12v_1v_3 + 2v_2v_8 + 8v_3^2 + 12v_3v_6 + 2v_4v_{10} - 24v_1y_5^2 - v_2y_7^2 - 20v_3y_5^2 - v_4y_7^2 - 24v_6y_5^2 - 4v_{10}y_6^2 + 8y_5^4 + 2y_6^2y_7^2) \quad \text{(A24)}$$

$$\frac{dv_4}{dl} = \varepsilon v_4 - K_4(12v_1v_4 + 2v_2v_9 + 2v_3v_{10} + 8v_4^2 + 12v_4v_7 - 24v_1y_6^2$$

$$- v_2y_7^2 - v_3y_7^2 - 20v_4y_6^2 - 24v_7y_6^2 - 4v_{10}y_5^2 + 2y_5^2y_7^2 + 8y_6^4) \qquad \text{(A25)}$$

$$\frac{dv_5}{dl} = \varepsilon v_5 - K_4\left(v_2^2 + 36v_5^2 + v_8^2 + v_9^2 - v_8y_7^2 - v_9y_7^2 + \frac{1}{4}y_7^4\right) \qquad \text{(A26)}$$

$$\frac{dv_6}{dl} = \varepsilon v_6 - K_4\left(v_3^2 + 36v_6^2 + v_8^2 + v_{10}^2 - 4v_3y_5^2\right.$$

$$\left. - 24v_6y_5^2 - v_8y_7^2 - v_{10}y_7^2 + \frac{1}{4}y_7^4 + 4y_5^4\right) \qquad \text{(A27)}$$

$$\frac{dv_7}{dl} = \varepsilon v_7 - K_4\left(v_4^2 + 36v_7^2 + v_9^2 + v_{10}^2 - 4v_4y_6^2\right.$$

$$\left. - 24v_7v_6^2 - v_9y_7^2 - v_{10}y_7^2 + 4y_6^4 + \frac{1}{4}y_7^4\right) \qquad \text{(A28)}$$

$$\frac{dv_8}{dl} = \varepsilon v_8 - K_4\left(2v_2v_3 + 12v_5v_8 + 12v_6v_8 + 8v_8^2 + 2v_9v_{10} - 6v_5y_7^2\right.$$

$$\left. - 6v_6y_7^2 - 4v_8y_7^2 - v_9y_7^2 - v_{10}y_7^2 - 4v_2y_5^2 - 4v_8y_5^2 + 2y_5^2y_7^2 + \frac{1}{2}y_7^4\right) \qquad \text{(A29)}$$

$$\frac{dv_9}{dl} = \varepsilon v_9 - K_4\left(2v_2v_4 + 12v_5v_9 + 12v_7v_9 + 2v_8v_{10} + 8v_9^2 - 4v_2y_6^2\right.$$

$$\left. - 6v_5y_7^2 - 6v_7y_7^2 - v_8y_7^2 - 4v_9y_6^2 - 4v_9y_7^2 - v_{10}y_7^2 + 2y_6^2y_7^2 + \frac{1}{2}y_7^4\right) \qquad \text{(A30)}$$

$$\frac{dv_{10}}{dl} = \varepsilon v_{10} - K_4\left(2v_3v_4 + 12v_6v_{10} + 12v_7v_{10} + 2v_8v_9 + 8v_{10}^2 - 4v_3y_6^2\right.$$

$$- 4v_4y_5^2 - 6v_6y_7^2 - 6v_7y_7^2 - v_8y_7^2 - v_9y_7^2 - 4v_{10}y_5^2 - 16v_{10}y_5y_6 - 4v_{10}y_6^2$$

$$\left. - 4v_{10}y_7^2 + 8y_5^2y_6^2 + 2y_5^2y_7^2 + 4y_5y_6y_7^2 + 2y_6^2y_7^2 + \frac{1}{2}y_7^4\right) \qquad \text{(A31)}$$

If we substitute the mean-field relations in the UB transition curve we find,

$$\frac{dv_1}{dl} = \frac{1}{2}\frac{dv_2}{dl} = \frac{dv_5}{dl}$$

$$\frac{dv_3}{dl} = \frac{dv_4}{dl} = \frac{dv_8}{dl} = \frac{dv_9}{dl}$$

$$\frac{dv_6}{dl} = \frac{dv_7}{dl} = \frac{1}{2}\frac{dv_{10}}{dl}$$

APPENDIX B

Here are the quadratic coefficients after we have integrated out φ_1, (where we have a compressible lattice),

$$\tau_2 = r_2 - \frac{2h(y_1 + y_8)}{3r_1 + 2x} \tag{B1}$$

$$\tau_3 = r_3 - \frac{2h(y_2 + y_9)}{3r_1 + 2x} \tag{B2}$$

$$\tau_4 = r_4 - \frac{2h(y_3 + y_{10})}{3r_1 + 2x} \tag{B3}$$

$$\tau_5 = r_5 - \frac{2h(y_4 + y_{11})}{3r_1 + 2x} \tag{B4}$$

The third order coefficients, y_5, y_6, and y_7 are unaffected by the integration over φ_1.

The next set of quartic coefficients are analogous to the incompressible case considered earlier,

$$v_1(q) = z_1 - \frac{1}{6}\frac{y_1^2}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B5}$$

$$v_2(q) = z_2 - \frac{1}{3}\frac{y_1y_2}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B6}$$

$$v_3(q) = z_3 - \frac{1}{3}\frac{y_1y_3}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B7}$$

$$v_4(q) = z_4 - \frac{1}{3}\frac{y_1y_4}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B8}$$

$$v_5(q) = z_5 - \frac{1}{6}\frac{y_2^2}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B9}$$

$$v_6(q) = z_6 - \frac{1}{6}\frac{y_3^2}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B10}$$

$$v_7(q) = z_7 - \frac{1}{6}\frac{y_4^2}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B11}$$

$$v_8(q) = z_8 - \frac{1}{3}\frac{y_2y_3}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B12}$$

$$v_9(q) = z_9 - \frac{1}{3}\frac{y_2y_4}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B13}$$

$$v_{10}(q) = z_{10} - \frac{1}{3}\frac{y_3y_4}{r_1 + q^2 + \frac{2}{3}x\delta(\mathbf{q})} \tag{B14}$$

The last set of quartic coefficients are associated with the infinite range energy density-energy density terms, so are momentum independent,

$$v_{11} = z_{11} - \frac{1}{6}\frac{y_8^2 + 2y_1y_8}{r_1 + \frac{2}{3}x} \tag{B15}$$

$$v_{12} = z_{12} - \frac{1}{3}\frac{y_1y_9 + y_2y_8 + y_8y_9}{r_1 + \frac{2}{3}x} \tag{B16}$$

$$v_{13} = z_{13} - \frac{1}{3}\frac{y_1y_{10} + y_3y_8 + y_8y_{10}}{r_1 + \frac{2}{3}x} \tag{B17}$$

$$v_{14} = z_{14} - \frac{1}{3}\frac{y_1y_{11} + y_4y_8 + y_8y_{11}}{r_1 + \frac{2}{3}x} \tag{B18}$$

$$v_{15} = z_{15} - \frac{1}{6}\frac{y_9^2 + 2y_2y_9}{r_1 + \frac{2}{3}x} \tag{B19}$$

$$v_{16} = z_{16} - \frac{1}{6}\frac{y_{10}^2 + 2y_3y_{10}}{r_1 + \frac{2}{3}x} \tag{B20}$$

$$v_{17} = z_{17} - \frac{1}{6}\frac{y_{11}^2 + 2y_4y_{11}}{r_1 + \frac{2}{3}x} \tag{B21}$$

$$v_{18} = z_{18} - \frac{1}{3}\frac{y_2y_{10} + y_3y_9 + y_9y_{10}}{r_1 + \frac{2}{3}x} \tag{B22}$$

$$v_{19} = z_{19} - \frac{1}{3}\frac{y_2y_{11} + y_4y_9 + y_9y_{11}}{r_1 + \frac{2}{3}x} \tag{B23}$$

$$v_{20} = z_{20} - \frac{1}{3}\frac{y_3y_{11} + y_4y_{10} + y_{10}y_{11}}{r_1 + \frac{2}{3}x} \tag{B24}$$

If we substitute the mean field values for the coefficients into the above expressions we see that the integration over φ_1 does not affect the relations among the coefficients, $\tau_2 = \tau_3$, $\tau_4 = \tau_5$, $v_1 = \frac{1}{2}v_2 = v_5$, $v_3 = v_4 = v_8 = v_9$, $v_6 = v_7 = \frac{1}{2}v_{10}$, $v_{11} = \frac{1}{2}v_{12} = v_{15}$, $v_{13} = v_{14} = v_{18} = v_{19}$, $v_{16} = v_{17} = \frac{1}{2}v_{20}$.

The differential recursion relations for the quadratic coefficients in the compressible case are given by the equations,

$$\frac{d\tau_2}{dl} = 2\tau_2 + K_4(12v_1(1 - \tau_2) + 2v_2(1 - \tau_3) + 2v_3(1 - \tau_4) + 2v_4(1 - \tau_5)$$
$$+ 4v_{11}(1 - \tau_2) + 2v_{12}(1 - \tau_3) + 2v_{13}(1 - \tau_4) + 2v_{14}(1 - \tau_5) - 2y_5^2 - 2y_6^2) \tag{B25}$$

$$\frac{d\tau_3}{dl} = 2\tau_3 + K_4(2v_2(1 - \tau_2) + 12v_5(1 - \tau_3) + 2v_8(1 - \tau_4) + 2v_9(1 - \tau_5)$$
$$+ 2v_{12}(1 - \tau_2) + 4v_{15}(1 - \tau_3) + 2v_{18}(1 - \tau_4) + 2v_{19}(1 - \tau_5) - y_7^2) \tag{B26}$$

$$\frac{d\tau_4}{dl} = 2\tau_4 + K_4(2v_3(1 - \tau_2) + 2v_8(1 - \tau_3) + 12v_6(1 - \tau_4) + 2v_{10}(1 - \tau_5)$$

$$+ 2v_{13}(1 - \tau_2) + 2v_{18}(1 - \tau_3) + 4v_{16}(1 - \tau_4) + 2v_{20}(1 - \tau_5) - 4y_5^2 - y_7^2) \tag{B27}$$

$$\frac{d\tau_5}{dl} = 2\tau_5 + K_4(2v_4(1 - \tau_2) + 2v_9(1 - \tau_3) + 2v_{10}(1 - \tau_4) + 12v_7(1 - \tau_5)$$

$$+ 2v_{14}(1 - \tau_2) + 2v_{19}(1 - \tau_3) + 2v_{20}(1 - \tau_4) + 4v_{17}(1 - \tau_5) - 4y_6^2 - y_7^2) \tag{B28}$$

Using the mean field relations, we find, $d\tau_2/dl = d\tau_3/dl$ and $d\tau_4/dl = d\tau_5/dl$.

The recursion relations for the third order coefficients are,

$$\frac{dy_5}{dl} = \left(1 + \frac{\varepsilon}{2}\right) y_5 - K_4(8v_3y_5 + 12v_6y_5 + 2v_{10}y_6 - 4y_5^3 - y_6y_7^2) \tag{B29}$$

$$\frac{dy_6}{dl} = \left(1 + \frac{\varepsilon}{2}\right) y_6 - K_4(8v_4y_6 + 12v_7y_6 + 2v_{10}y_5 - y_5y_7^2 - 4y_6^3) \tag{B30}$$

$$\frac{dy_7}{dl} = \left(1 + \frac{\varepsilon}{2}\right) y_7 - K_4(4v_8y_7 + 4v_9y_7 + 4v_{10}y_7 - 4y_5y_6y_7 - y_7^3) \tag{B31}$$

These differential recursion relations satisfy the equalities, $dy_5/dl = -dy_6/dl = \frac{1}{2}dy_7/dl$, on the uniaxial-biaxial transition curve.

The first set of fourth order coefficients are,

$$\frac{dv_1}{dl} = \varepsilon v_1 - K_4(36v_1^2 + v_2^2 + v_3^2 + v_4^2 - 4v_3y_5^2 - 4v_4y_6^2 + 2y_5^4 + 2y_6^4) \tag{B32}$$

$$\frac{dv_2}{dl} = \varepsilon v_2 - K_4(12v_1v_2 + 8v_2^2 + 12v_2v_5 + 2v_3v_8 + 2v_4v_9 - v_3y_7^2$$

$$- v_4y_7^2 - 4v_8y_5^2 - 4v_9y_6^2 + 2y_5^2y_7^2 + 2y_5y_6y_7^2 + 2y_6^2y_7^2) \tag{B33}$$

$$\frac{dv_3}{dl} = \varepsilon v_3 - K_4(12v_1v_3 + 2v_2v_8 + 8v_3^2 + 12v_3v_6 + 2v_4v_{10} - 24v_1y_5^2$$

$$- v_2y_7^2 - 4v_3y_5^2 - 16v_3y_5^2 - v_4y_7^2 - 24v_6y_5^2 - 4v_{10}y_6^2 + 8y_5^4 + 2y_6^2y_7^2) \tag{B34}$$

$$\frac{dv_4}{dl} = \varepsilon v_4 - K_4(12v_1v_4 + 2v_2v_9 + 2v_3v_{10} + 8v_4^2 + 12v_4v_7 - 24v_1y_6^2$$

$$- v_2y_7^2 - v_3y_7^2 - 20v_4y_6^2 - 24v_7y_6^2 - 4v_{10}y_5^2 + 2y_5^2y_7^2 + 8y_6^4) \tag{B35}$$

$$\frac{dv_5}{dl} = \varepsilon v_5 - K_4\left(v_2^2 + 36v_5^2 + v_8^2 + v_9^2 - v_8y_7^2 - v_9y_7^2 + \frac{1}{4}y_7^4\right) \tag{B36}$$

$$\frac{dv_6}{dl} = \varepsilon v_6 - K_4\Big(v_3^2 + 36v_6^2 + v_8^2 + v_{10}^2 - 4v_3y_5^2$$

$$- 24v_6y_5^2 - v_8y_7^2 - v_{10}y_7^2 + 4y_5^4 + \frac{1}{4}y_7^4\Big) \quad \text{(B37)}$$

$$\frac{dv_7}{dl} = \varepsilon v_7 - K_4\Big(v_4^2 + 36v_7^2 + v_9^2 + v_{10}^2 - 4v_4y_6^2$$

$$- 24v_7y_6^2 - v_9y_7^2 - v_{10}y_7^2 + 4y_6^4 + \frac{1}{4}y_7^4\Big) \quad \text{(B38)}$$

$$\frac{dv_8}{dl} = \varepsilon v_8 - K_4\Big(2v_2v_3 + 12v_5v_8 + 12v_6v_8 + 8v_8^2 + 2v_9v_{10} - 4v_2y_5^2$$

$$- 6v_5y_7^2 - 6v_6y_7^2 - 4v_8y_5^2 - 4v_8y_7^2 - v_9y_7^2 - v_{10}y_7^2 + 2y_5^2y_7^2 + \frac{1}{2}y_7^4\Big) \quad \text{(B39)}$$

$$\frac{dv_9}{dl} = \varepsilon v_9 - K_4\Big(2v_2v_4 + 12v_5v_9 + 12v_7v_9 + 2v_8v_{10} + 8v_9^2 - 4v_2y_6^2$$

$$- 6v_5y_7^2 - 6v_7y_7^2 - v_8y_7^2 - 4v_9y_6^2 - 4v_9y_7^2 - v_{10}y_7^2 + 2y_6^2y_7^2 + \frac{1}{2}y_7^4\Big) \quad \text{(B40)}$$

$$\frac{dv_{10}}{dl} = \varepsilon v_{10} - K_4\Big(2v_3v_4 + 12v_6v_{10} + 12v_7v_{10} + 2v_8v_9 + 8v_{10}^2 - 4v_3y_6^2$$

$$- 4v_4y_5^2 - 6v_6y_7^2 - 6v_7y_7^2 - v_8y_7^2 - v_9y_7^2 - 4v_{10}y_5^2 - 16v_{10}y_5y_6$$

$$- 4v_{10}y_6^2 - 4v_{10}y_7^2 + 8y_5^2y_6^2 + 2y_5^2y_7^2 + 4y_5y_6y_7^2 + 2y_6^2y_7^2 + \frac{1}{2}y_7^4\Big) \quad \text{(B41)}$$

On the UB transition curve,

$$\frac{dv_1}{dl} = \frac{1}{2}\frac{dv_2}{dl} = \frac{dv_5}{dl}$$

$$\frac{dv_3}{dl} = \frac{dv_4}{dl} = \frac{dv_8}{dl} = \frac{dv_9}{dl}$$

$$\frac{dv_6}{dl} = \frac{dv_7}{dl} = \frac{1}{2}\frac{dv_{10}}{dl}$$

The last set of recursion relations for the fourth order infinite range coupling coefficients are,

$$\frac{dv_{11}}{dl} = \varepsilon v_{11} - K_4(24v_1v_{11} + 2v_2v_{12} + 2v_3v_{13} + 2v_4v_{14} + 4v_{11}^2 + v_{12}^2 + v_{13}^2 + v_{14}^2 - 4v_{13}y_5^2 - 4v_{14}y_6^2) \quad \text{(B42)}$$

$$\frac{dv_{12}}{dl} = \varepsilon v_{12} - K_4(12v_1v_{12} + 4v_2v_{11} + 4v_2v_{15} + 2v_3v_{18} + 2v_4v_{19} + 12v_5v_{12} + 2v_8v_{13} + 2v_9v_{14} + 4v_{11}v_{12} + 4v_{12}v_{15} + 2v_{13}v_{18} + 2v_{14}v_{19} - v_{13}y_7^2 - v_{14}y_7^2 - 4v_{18}y_5^2 - 4v_{19}y_6^2) \quad \text{(B43)}$$

$$\frac{dv_{13}}{dl} = \varepsilon v_{13} - K_4(12v_1v_{13} + 2v_2v_{18} + 4v_3v_{11} + 4v_3v_{16} + 2v_4v_{20} + 12v_6v_{13} + 2v_8v_{12} + 2v_{10}v_{14} + 4v_{11}v_{13} + 2v_{12}v_{18} + 4v_{13}v_{16} + 2v_{14}v_{20} - 8v_{11}y_5^2 - v_{12}y_7^2 - 4v_{13}y_5^2 - v_{14}y_7^2 - 8v_{16}y_5^2 - 4v_{20}y_6^2) \quad \text{(B44)}$$

$$\frac{dv_{14}}{dl} = \varepsilon v_{14} - K_4(12v_1v_{14} + 2v_2v_{19} + 2v_3v_{20} + 4v_4v_{11} + 4v_4v_{17} + 12v_7v_{14} + 2v_9v_{12} + 2v_{10}v_{13} + 4v_{11}v_{14} + 2v_{12}v_{19} + 2v_{13}v_{20} + 4v_{14}v_{17} - 8v_{11}y_6^2 - v_{12}y_7^2 - v_{13}y_7^2 - 4v_{14}y_6^2 - 8v_{17}y_6^2 - 4v_{20}y_5^2) \quad \text{(B45)}$$

$$\frac{dv_{15}}{dl} = \varepsilon v_{15} - K_4(2v_2v_{12} + 24v_5v_{15} + 2v_8v_{18} + 2v_9v_{19} + v_{12}^2 + 4v_{15}^2 + v_{18}^2 + v_{19}^2 - v_{18}y_7^2 - v_{19}y_7^2) \quad \text{(B46)}$$

$$\frac{dv_{16}}{dl} = \varepsilon v_{16} - K_4(2v_3v_{13} + 24v_6v_{16} + 2v_8v_{18} + 2v_{10}v_{20} + v_{13}^2 + 4v_{16}^2 + v_{18}^2 + v_{20}^2 - 4v_{13}y_5^2 - 8v_{16}y_5^2 - v_{18}y_7^2 - v_{20}y_7^2) \quad \text{(B47)}$$

$$\frac{dv_{17}}{dl} = \varepsilon v_{17} - K_4(2v_4v_{14} + 24v_7v_{17} + 2v_9v_{19} + 2v_{10}v_{20} + v_{14}^2 + 4v_{17}^2 + v_{19}^2 + v_{20}^2 - 4v_{14}y_6^2 - 8v_{17}y_6^2 - v_{19}y_7^2 - v_{20}y_7^2) \quad \text{(B48)}$$

$$\frac{dv_{18}}{dl} = \varepsilon v_{18} - K_4(2v_2v_{13} + 2v_3v_{12} + 12v_5v_{18} + 12v_6v_{18} + 4v_8v_{15} + 4v_8v_{16}$$

$$+ 2v_9v_{20} + 2v_{10}v_{19} + 2v_{12}v_{13} + 4v_{15}v_{18} + 4v_{16}v_{18} + 2v_{19}v_{20} - 4v_{12}y_5^2$$
$$- 2v_{15}y_7^2 - 2v_{16}y_7^2 - v_{19}y_7^2 - v_{20}y_7^2 - 4v_{18}y_5^2) \quad \text{(B49)}$$

$$\frac{dv_{19}}{dl} = \varepsilon v_{19} - K_4(2v_2v_{14} + 2v_4v_{12} + 12v_5v_{19} + 12v_7v_{19} + 2v_8v_{20}$$
$$+ 4v_9v_{15} + 4v_9v_{17} + 2v_{10}v_{18} + 2v_{12}v_{14} + 4v_{15}v_{19} + 4v_{17}v_{19} + 2v_{18}v_{20}$$
$$- 4v_{12}y_6^2 - 2v_{15}y_7^2 - 2v_{17}y_7^2 - v_{18}y_7^2 - 4v_{19}y_6^2 - v_{20}y_7^2) \quad \text{(B50)}$$

$$\frac{dv_{20}}{dl} = \varepsilon v_{20} - K_4(2v_3v_{14} + 2v_4v_{13} + 12v_6v_{20} + 12v_7v_{20} + 2v_8v_{19} + 2v_9v_{18}$$
$$+ 4v_{10}v_{16} + 4v_{10}v_{17} + 2v_{13}v_{14} + 4v_{16}v_{20} + 4v_{17}v_{20} + 2v_{18}v_{19} - 4v_{13}y_6^2$$
$$- 4v_{14}y_5^2 - 2v_{16}y_7^2 - 2v_{17}y_7^2 - v_{18}y_7^2 - v_{19}y_7^2 - 4v_{20}y_5^2 - 4v_{20}y_6^2) \quad \text{(B51)}$$

These relations obey the same type of equalities as the first set of fourth order coefficients on the UB transition,

$$\frac{dv_{11}}{dl} = \frac{1}{2}\frac{dv_{12}}{dl} = \frac{dv_{15}}{dl}$$

$$\frac{dv_{13}}{dl} = \frac{dv_{14}}{dl} = \frac{dv_{18}}{dl} = \frac{dv_{19}}{dl}$$

$$\frac{dv_{16}}{dl} = \frac{dv_{17}}{dl} = \frac{1}{2}\frac{dv_{20}}{dl}$$

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